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# Reconciling TEV and VaR in Active Portfolio Management: A New Frontier

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**Abstract**

This article investigates the risk-return relationship of managed portfolios when two risk indicators, the Tracking Error Volatility (TEV) and the Value-at-Risk (VaR), are both constrained not to exceed pre-set maximum values. While in some cases these constraints may not be mutually compatible, it is often possible to find portfolios that satisfy both constraints. In this paper, we analyze the problem of choosing among these.

Focusing on the trade-off between the joint restrictions that can be imposed on both risk indicators, we define the Risk Balancing Frontier (RBF), a new portfolio boundary in the traditional absolute risk-total return space, that contains all the portfolios characterized by the minimum VaR attainable for each TEV level. We show that the RBF is the set of all tangency portfolios between two well-known frontiers: the so-called Constrained Tracking Error Volatility Frontier (Jorion, 2003) and the Constrained Value-at-Risk Frontier (Alexander and Baptista, 2008). Thus, the RBF is useful for analyzing the agency problem in delegated portfolio management.

The RBF does not have a closed-form definition and must be determined numerically: to this aim, we develop a fast and accurate algorithm.

**JEL Class.:** C61, G11

**Keywords:** Benchmarking, portfolio frontiers, tracking error volatility, Value-at-Risk, misalignment of objectives

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# Reconciling TEV and VaR in Active Portfolio Management: A New Frontier

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## 1 Introduction

In contemporary financial markets, most investors delegate their investment decisions to professionals and establish agency relationships, whose impact on financial markets and on economic development at the macro level is considered very important by the literature (Stracca, 2006). Therefore, an accurate analysis of the risk-return relationship in modern financial markets must take into account the issues of delegated portfolio management, when the objectives of the investor (the principal) and of the asset manager (the agent), are not perfectly aligned.

In this context, the literature is oriented, on the one hand, to the determination of optimal contracts between the portfolio manager and the investor that rewards performance relative to a benchmark, and, on the other hand, to the link between the manager compensation package and their behavior with respect to risk exposure (Ingan and Pinheiro, 2015). This article addresses the constraints imposed by investors on their agents.

As explained by Chow (1995), investors who choose a mutual fund with an active management strategy do so to maximize their utility, which is a function of portfolio return, variance and tracking error. Since returns are uncertain, Chow argues that investors may decide to compare the performance against a benchmark, but they are "still concerned with the prospect of losing money"; therefore, they "seek portfolios with high return, low standard deviation and low tracking error". Empirical evidence supports this idea, as most practitioners employ both total and relative risk measures. Consequently, in this paper, we will focus on both total and relative risk measures and, in particular, on active management aimed at maximizing the investor's utility.

One of the constraints most commonly used in portfolio management is to limit the asset manager's activity by setting a maximum value on the tracking error volatility (TEV), where the tracking error is the difference between the portfolio and the benchmark return. Thus, the portfolio risk is kept closer to that of the selected benchmark. Another possible constraint on the activity of asset manager is a limit to the Value-at-Risk (VaR), the most widely used absolute risk measure, in order to keep the maximum possible loss below a certain threshold with a given probability.

As is well known, the active approach aims to outperform a benchmark portfolio, while the goal of the passive approach is to match the performance of the benchmark;[1] the latter approach is exemplified by index funds of the exchanged traded funds (ETF) that replicate the benchmark's composition and performance.

Financial literature has documented the partial shift from active to passive management during the past decades in the wealth management market (see, for instance, Anadu, Kruttli, McCabe and Osambela, 2020). The growth of ETFs and passive management is due to many causes, but the two (arguably) most important reasons are high fees and under-performance by many mutual and hedge funds. The former is in fact correlated to the latter: when fees are high, the asset manager needs higher and higher returns in order to compensate the fee and to obtain a further gain for investors. The latter could be due by various factors such as increased market efficiency, lack of skills by some asset managers, or inadequate restrictions imposed by risk management offices.[2] The shift from active to passive investing has raised several concerns in the literature about the stability of financial markets (see *e. g.* Bolla, Kohler and Wittig, 2016; Ben-David, Franzoni and Moussawi, 2018; Bhattacharya and O'Hara, 2018). Anadu, Kruttli, McCabe and Osambela (2020) find that passive strategies may dampen liquidity and redemption risks, but can also amplify market volatility and increase industry concentration.

The growth in passive management, however, has not displaced active management strategies completely. Moreover, ETFs are themselves being used more and more often as a tool for active managment, since two-thirds of the new ETFs issued in the first six months of 2021 were active funds.[3] In fact, a close analysis on longer time periods reveals that there is no clear winner in active versus passive investing, but their relative performance is a cyclical phenomenon[4]. Moreover, the economic

[1] Some literature, however, has tried to reconcile the two approaches arguing that in practice a combination of the two may improve overall portfolio results. See for example Flood and Ramachandran (2000).

[2] According to Morningstar's Active/Passive Barometer (Johnson, 2019), 48% of active U.S. stock funds survived and outperformed their average passive peer over the 12 months through June 2019, up from 37% in the previous year, but "in general, actively managed funds have failed to survive and beat their benchmarks, especially over longer time horizons" and "the cheapest funds succeeded more than twice as often as the priciest ones". Smith (2017) states "Endowments and foundations have turned to passive investments after hedge funds disappointed with high fees and poor performance".

[3] See `https://www.nasdaq.com/articles/active-funds-are-dominating-2021-etf-launches-2021-07-14`.

[4] See `https://www.hartfordfunds.com/insights/market-perspectives/equity/cyclical-nature-active-passive-investing.html`.

recovery after the Covid-19 pandemic is based on fiscal and monetary stimuli that will probably determine a return to active management. The active investment will arguably be the most appropriate in the context of increasing inflation rates and subsidies targeted to industries that are currently under-represented in passively managed portfolios.

Therefore, we feel that new contributions to the existing literature on active portfolio management are necessary, especially with a view to determine strategies for improving portfolio performance when the investor-agent relationship are taken into account. This article analyzes those situations in which asset managers face a joint constraint on the TEV and the VaR. Although the existence of a portfolio consistent with both constraints is generally not guaranteed, a non-empty set of portfolios that satisfy both exists (see Palomba and Riccetti, 2012), when the constraints are not too stringent. In these cases, a trade-off emerges between the minimization of the value of the two risk indicators. Our focus is on those feasible portfolios for which the TEV and the VaR are lower than their pre-set maximum values.

More generally, we frame the problem of portfolio choice by considering the different objectives of the investor and of the asset manager due to their different risk appetite, a situation frequently met in delegated portfolio management. We classify portfolios on the basis of the following criterion: if a portfolio $P$ can be modified so as to improve the utility of one party without damaging the other and to remain feasible at the same time, then it is clearly sub-optimal in terms of risk-return. In this paper, we concentrate on the set $\mathbb{S}$ of portfolios that satisfy the following criterion: if $P \in \mathbb{S}$, no other portfolios exist in a neighborhood of $P$ such that the objective functions of both parties increase.

Thus, from the perspective of economic theory, the set $\mathbb{S}$ could be considered as a set of "Pareto-efficient" portfolios. The entire analysis is performed in the usual $(\sigma_P, \mu_P)$ space, where $\sigma_P$ and $\mu_P$ are absolute risk (standard deviation of the returns) and the expected total return of portfolio $P$. To be specific, we define a new portfolio frontier that contains all portfolios for which the VaR constraint is minimized for each TEV level. Therefore, this boundary can be seen as a set of equilibria that necessarily includes the benchmark, where the TEV equals zero by definition, as a special case.

We therefore consider the issue of keeping the two types of risk under control, while choosing efficient combinations in terms of portfolio risk and return. This is consistent with the theoretical framework developed by Jorion (2003) and the actual practice in the asset management industry. In this paper, we argue that a proper strategy for taking into account different risk measures jointly may significantly mitigate such shortcomings.

The remainder of this article proceeds as follows: after a literature review in section 2, we introduce in Section 3 our new portfolio boundary, and we discusse its properties and financial implications. In this context, we also present the numerical method for representing it as a curve in the $(\sigma_P, \mu_P)$ space. Section 4 provides a short empirical example, and finally Section 5 concludes.

# 2 Literature review

Several authors have proposed strategies to improve the portfolio performance relative to market performance (an index, or a naïve portfolio) and carry out horse races among portfolios. One of the most prominent papers in this literature is DeMiguel, Garlappi and Uppal (2009). Other relevant papers are, for example, Tu and Zhou (2011) who improve significantly on the performance of the equally weighted portfolio via optimal combinations of known portfolios (more precisely, those defined by Markowitz, 1952; Jorion, 1996; MacKinlay and Pástor, 2000; Kan and Zhou, 2007), while Jiang, Du, An and Zhang (2021) propose an allocation strategy that outperforms the naïve portfolio in terms of absolute risk and Sharpe ratio.

The issue of the possible inadequate restrictions imposed by the risk management office is analyzed by Riccetti (2012, 2017). As Riccetti (2012) argues, asset managers often follow their benchmarks passively, even if they receive a fee from investors to be active and beat the benchmark; this behavior could also be caused by risk managers who set a low fixed level of maximum TEV, often making the benchmark difficult to beat. To overcome this problem, the proposed solution is applying a lower limit on the TEV so as to force the asset manager to follow an active strategy. The same analysis is extended in Riccetti (2017): in order to help asset managers to (i) keep the risk of the portfolio (relatively) close to that of the selected benchmark, and (ii) beat the benchmark and maximize the investors' utility moving away from the benchmark, it is suggested that risk management offices should set consistent limits on TEV and VaR.

The analysis of the relationship between various risk measures and portfolio efficiency has traditionally been undertaken by considering geometrical objects in the $(\sigma_P, \mu_P)$ space. The Mean-Variance frontier (MVF) introduced by Markowitz (1952) is the cornerstone for defining other portfolio frontiers in the $(\sigma_P, \mu_P)$ space. The MVF is also crucial because it divides the plane into two regions, and identifies the surface area at its left as the set of inadmissible portfolios. Another relevant boundary is the Mean-TEV frontier (MTF), introduced by Roll (1992). This corresponds to a horizontal translation of the MVF since it is derived by minimizing the TEV rather than the portfolio variance. An important feature is that the benchmark portfolio necessarily lies on it.

Considering the constraints of maximum TEV and VaR, two other portfolio frontiers play a key role. Jorion (2003) introduces the Constrained TEV Frontier

(CTF), an elliptic portfolio frontier in the mean-variance space which takes an egg-like shape in the $(\sigma_P, \mu_P)$ space. The interesting feature is that the maximum TEV delimits a closed and bounded set of feasible portfolios that lie around the benchmark; the more stringent the TEV becomes, the more this boundary narrows around the benchmark itself. Alexander and Baptista (2008) define the Constrained VaR Frontier (CVF) as a positive-slope linear portfolio boundary in the $(\sigma_P, \mu_P)$ space, to the left of which all portfolios satisfy the VaR constraint. Since the vertical intercept of this function is equal to the negative of the VaR limit, the intercept gets higher as the constraint becomes tighter. These two frontiers are the starting point for our analysis, and we will use them for illustrating the compatibility issues for the TEV and the VaR constraints in Section 3. Building on previous work by Alexander and Baptista (2008), Palomba and Riccetti (2012) analyzed the space of feasible portfolios that satisfy both TEV and VaR constraints, and introduced the Fixed VaR-TEV Frontier (FVTF), thus obtaining various scenarios according to the predetermined values assigned to the two risk indicators.[5]

The literature on the relationships between different portfolio frontiers has subsequently developed in several directions. For example, Alexander and Baptista (2010) propose a strategy for active portfolio management in which they introduce a new portfolio frontier that contains all portfolios which minimize the TEV for any given *ex ante* portfolio alpha, where alpha is the intercept of the linear regression of the portfolio return on the benchmark return. This approach provides a new viewpoint about the active management strategies and identifies portfolios that simultaneously satisfy more than one criterion. In this context, Stucchi (2015) studies the relationships between the contributions of Roll (1992), Jorion (2003), Alexander and Baptista (2008, 2010) and Palomba and Riccetti (2012), and proposes a unified approach in order to summarize their results into a single optimal allocation strategy that works under different additional risk constraints.

Other authors consider portfolio performance under simultaneous TEV and weight constraints compliance, starting from the contribution of Bajeux-Besnainou, Belhaj, Maillard and Portait (2011) up to the work of Daly and Van Vuuren (2020) in which new constrained portfolio frontiers are defined. Recently, Du Sart and Van Vuuren (2021) focus on two portfolios lying on the Jorion's CTF and analyse their composition and performance in comparison with the maximum Sharpe Ratio portfolio. Using data from South Africa, they illustrate how these portfolios perform during bull and bear markets.

Moreover, a large number of contributions put forward enhancements of Jorion's original proposal by introducing constraints on different quantities (see, for instance, Ammann and Zimmermann, 2001; El-Hassan and Kofman, 2003; Maxwell, Daly, Thomson and Van Vuuren, 2018; Maxwell and Van Vuuren, 2019). Bertrand (2010)

[5]A graphical representation of these objects is given Figure D.1 in the appendix.

proposes an alternative to Jorion's approach by introducing the Constant Risk Aversion frontier. This boundary is based on the risk aversion parameter, defined as the marginal rate of substitution between the portfolio variance and expected return. Another recent contribution is provided by Stowe (2019), who reformulates the models by Best and Grauer (1990) and Jorion (2003) by considering the maximization of a quadratic utility function under several combinations of linear and quadratic constraints that correspond to different restrictions on the portfolio expected return or on the TEV.

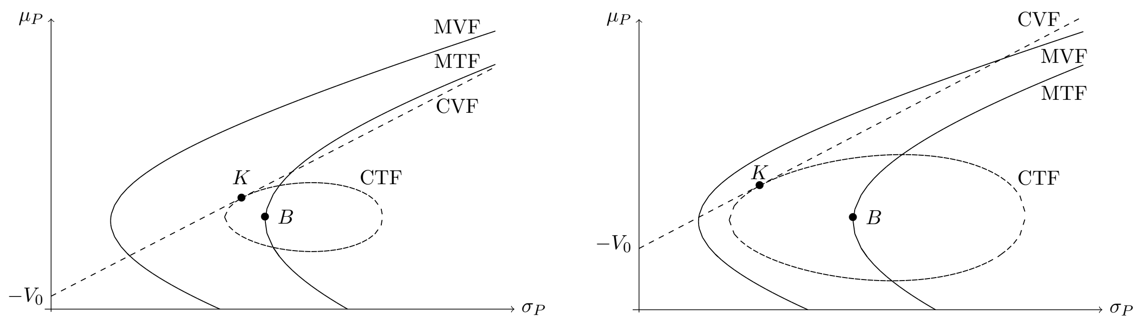
Finally, Palomba and Riccetti (2019) focus on the portfolio efficiency issue when restriction to TEV, VaR and possibly to the overall variance are jointly set. The main result is the formal definition of some portfolio frontiers that satisfy all the restrictions on risk indicators and contain only non-dominated portfolios in terms of variance and return.

# 3 Minimum VaR given a maximum TEV: a new portfolio frontier

As we remarked earlier, the TEV and the VaR constraints may be binding, or not. Consider Figure 1, where Jorion's CTF and Alexander and Baptista's CVF are depicted.

Figure 1: Tangency portfolio $K$ (TEV $= T_0$ and VaR $= V_0$)

(a) stringent TEV, weak VaR restriction (b) stringent VaR, weak TEV restriction




**Legend:** MVF: Mean-Variance Frontier; MTF: Mean-TEV Frontier; CVF: Constrained VaR frontier; $B$: Benchmark portfolio; $K$: Tangency portfolio; $V_0$: Target VaR.

The case of incompatible restrictions arises when the oval boundary lies completely to the right of the linear one, and there are no intersections. Otherwise, their intersection contains all portfolios that jointly satisfy the inequalities TEV $\leq T_0$ and

VaR $\leq V_0$, where $T_0$ and $V_0$ are the constraints faced by asset managers. If the CTF and the CVF are tangent, a unique portfolio is available for the asset manager, and strict equality holds for both restriction TEV $= T_0$ and VaR $= V_0$. In this case, we define this unique portfolio $K \equiv (\sigma_K, \mu_K)$ as the tangency portfolio, while we use the symbol $B$ to indicate the benchmark the TEV is computed against.

The scenario in Figure 1(a) occurs when the restrictions are mainly aimed at reducing relative risk, so that the constraint on the VaR of the portfolio $K$ is not particularly severe. Conversely, in the scenario in Figure 1(b) the maximum TEV is larger, so the VaR limit on $K$ is more binding. The eccentricity of the CTF and the intercept of the CVF can be taken as graphical hints on how stringent the constraints on the TEV and on the VaR are. Since the portion of the plane delimited by the CTF increases with $T_0$, it is apparent from Figure 1 that the position of portfolio $K$ depends on both $T_0$ and $V_0$, with a trade-off between the two.

Since our aim is to define a new portfolio boundary in the traditional $(\sigma_P, \mu_P)$ space, we identify a subset of efficient VaR-TEV portfolios such that the VaR is minimized for a given TEV. This subset enjoys a Pareto property: if portfolios are chosen on the preferences of the parties, and these can be represented by the TEV and the VaR, choices outside the subset would be sub-optimal. The tangency portfolio $K$ in Figure 1 possesses exactly all these characteristics.

## 3.1 The Risk Balancing Frontier (RBF)

In our analysis, we use the same setup as in Alexander and Baptista (2008) and Palomba and Riccetti (2012). We assume that the parties can choose among $n$ risky assets, with $\boldsymbol{\mu}$ being the $n$-dimensional column vector of expected returns, and $\boldsymbol{\Sigma}$ their covariance matrix, which we assume nonsingular. We define the parameters $a = \boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}$, $b = \boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and $c = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, where $\boldsymbol{\iota} = \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}'$ is an $n$-dimensional column vector. We also define $\boldsymbol{\omega_C}$ as the 'Global Minimum Variance' portfolio and $C$ as the corresponding point on the $(\sigma_P, \mu_P)$ plane; $\boldsymbol{\omega_C}$ has expected return $\mu_C = b/a$ and $\sigma_C^2 = 1/a$, while $\mu_B$ and $\sigma_B^2$ are the benchmark return and variance.

We make the customary assumptions of unlimited short sales, quadratic utility function and/or normally distributed returns; these assumptions rule out skewed and leptokurtic return distributions, so that the portfolio standard deviation is the unique risk factor. The analytical definition of the CTF and CVF boundaries in the $(\sigma_P, \mu_P)$ space are provided by equations

$$d(\sigma_P^2 - \sigma_B^2 - T_0)^2 + 4\Delta_2(\mu_P - \mu_B)^2 - 4\Delta_1(\sigma_P^2 - \sigma_B^2 - T_0)(\mu_P - \mu_B) - 4d\delta_B T_0 = 0 \quad (1)$$

$$\mu_P = z_\theta \sigma_P - V_0, \quad (2)$$

where $d = c - b^2/a$, $\Delta_1 = \mu_B - \mu_C$, $\Delta_2 = \sigma_B^2 - \sigma_C^2$, $\delta_B = \Delta_2 - \Delta_1/d$; $z_\theta$ is the standard normal quantile (with $0.5 \leq \theta < 1$) and $T_0$ and $V_0$ are the constraints set on TEV and VaR. Equation (1) was introduced by Jorion (2003), and identifies a set of points in $(\sigma_P, \mu_P)$ space that takes a (somewhat distorted) oval shape. Equation (2), instead, was put forward in Alexander and Baptista (2008) and produces an upward-sloped straight line.

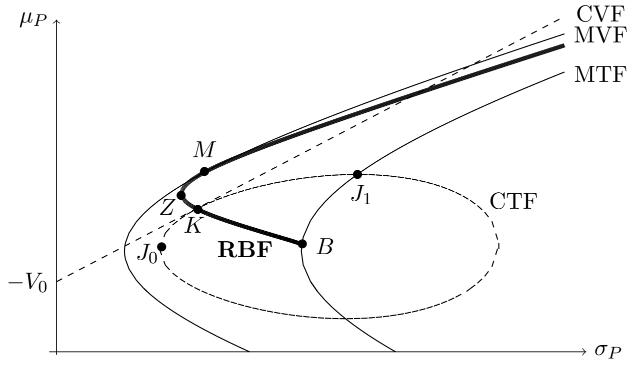
Given the distribution of the portfolio returns, the VaR is $V_P = z_\theta \sigma_P - \mu_P$. Since $\sqrt{d}$ is the asymptotic slope of the Markowitz Mean-Variance frontier, two distinct cases arise: the high confidence case, for which the VaR line has a steeper slope than the asymptotic slope of the MVF ($z_\theta > \sqrt{d}$), and a low confidence level, in the opposite case.

For any portfolio $P \in (\sigma_P, \mu_P)$, define $T(P)$ as the TEV of $P$ with respect to the chosen benchmark $B$ and $V(P, \theta) = z_\theta \sigma_P - \mu_P$ its VaR for a given risk level $\theta$. Now consider the portfolio that minimizes the VaR subject to a given TEV $= T_0$ given by

$$\hat{P}(T_0, \theta) = \underset{T(P)=T_0}{\operatorname{argmin}} V(P, \theta); \tag{3}$$

we define the Risk Balancing Frontier (RBF from here on) as the subset of $(\sigma_P, \mu_P)$ space containing all portfolios $\hat{P}(T_0, \theta)$ for $T_0 \in (0, T^{\max})$; the bold line plotted in Figure 2 provides a graphical example. The rest of the paper will focus on the case $z_\theta > \sqrt{d}$ (high confidence case), that we consider the most realistic, in the light of the fact that risk management offices customarily set $\theta$ very close to 1. Note also that the minimization of $V(P, \theta)$ is defined under the constraint $T(P) = T_0$; in fact, it may be more realistic to consider the weak-inequality constraint $T(P) \leq T_0$. This issue will be analyzed in Section 3.3.

Figure 2: The Risk Balancing Frontier (RBF)




From an analytical point of view, the RBF corresponds to a specific *locus*, entirely contained within the Mean-Variance frontier where the equality $\text{CTF}(\sigma_P^2, \mu_P; T_0) = \text{CVF}(\sigma_P^2, \mu_P; V_0)$ holds and the VaR for each portfolio is the minimum attainable for a given TEV $= T_0$. The RBF identifies a continuous set of points in the $(\sigma_P, \mu_P)$

space; this property derives from the RBF being an envelope of several optima under a continuously-varying constraint. It can therefore be thought of as the set of portfolios that correspond to all possible risk-return space coordinates of the tangency portfolio $K$ for increasing levels of $T_0$ (see Figure 2).

Along this path, two notable points can be identified:

**the portfolio Z,** defined as the portfolio in which the variance is minimized;

**the portfolio M,** defined as the portfolio for which the efficiency loss[6] is zero. $M$ corresponds to the contact point with the MVF, and minimizes the VaR among all admissible portfolios.

The definition of the RBF implies that its position in the $(\sigma_P, \mu_P)$ space depends on the benchmark coordinates, as well as the location of the zero efficiency loss portfolio $M$. The boundary shape is independent of the slope of the CVF, $z_\theta$, and that of the horizontal axis of the CTF, $\Delta_1$.

In order to analyze the properties of the whole set of portfolios, however, it is convenient to define formally the RBF as the solution to a constrained optimization problem, which is what we do in the next subsection.

## 3.2 Derivation of the RBF

In order to find an explicit solution to equation (3), we re-state the optimization problem as

$$\begin{array}{lll} \max & -\text{VaR} = z_\theta \sigma_P(\boldsymbol{\omega}) - \mu_P(\boldsymbol{\omega}) & \\ \text{sub} & \sqrt{\text{TEV}} = \sqrt{T_0} & \Rightarrow \\ & \text{fully invested portfolio} & \end{array} \quad \begin{array}{ll} \min & \text{VaR} = z_\theta \sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}} - \boldsymbol{\omega}'\boldsymbol{\mu} \\ \text{sub} & \sqrt{(\boldsymbol{\omega} - \boldsymbol{\omega_B})'\boldsymbol{\Sigma}(\boldsymbol{\omega} - \boldsymbol{\omega_B})} = \sqrt{T_0} \\ & \boldsymbol{\omega}'\boldsymbol{\iota} = 1, \end{array} \tag{4}$$

where $\boldsymbol{\omega_B}$ is the benchmark portfolio. Equation (4) leads to the following Lagrangian:

$$\mathcal{L}(\boldsymbol{\omega}, T_0) = z_\theta \sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}} - \boldsymbol{\omega}'\boldsymbol{\mu} - \lambda_1 [\sqrt{(\boldsymbol{\omega} - \boldsymbol{\omega_B})'\boldsymbol{\Sigma}(\boldsymbol{\omega} - \boldsymbol{\omega_B})} - \sqrt{T_0}] - \lambda_2 [\boldsymbol{\omega}'\boldsymbol{\iota} - 1], \tag{5}$$

where the scalars $\lambda_1$ and $\lambda_2$ are the shadow prices. For the first order conditions we get

$$\boldsymbol{\nabla}(\boldsymbol{\omega}, T_0) = r(\boldsymbol{\omega}, \theta)\boldsymbol{\Sigma}\boldsymbol{\omega} - \boldsymbol{\mu} - \lambda_1 \frac{1}{\sqrt{T_0}} \boldsymbol{\Sigma}(\boldsymbol{\omega} - \boldsymbol{\omega_B}) - \lambda_2 \boldsymbol{\iota} = \mathbf{0} \tag{6}$$

[6]We use the customary definition of efficiency loss as the horizontal distance in the $(\sigma_P, \mu_P)$ space between a portfolio and the Markowitz frontier MVF.

where $r(\boldsymbol{\omega},\theta) = \dfrac{z_\theta}{\sigma_P(\boldsymbol{\omega})}$ is a strictly positive scalar function; equation (6) may be transformed by premultiplying $\boldsymbol{\nabla}(\boldsymbol{\omega}, T_0)$ by the inverse of $\boldsymbol{\Sigma}$,

$$\boldsymbol{\nabla}^*(\boldsymbol{\omega}, T_0) = r(\boldsymbol{\omega},\theta)\boldsymbol{\omega} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \lambda_1 \frac{1}{\sqrt{T_0}}(\boldsymbol{\omega} - \boldsymbol{\omega_B}) - \lambda_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\iota} = \mathbf{0}; \tag{7}$$

therefore, the solutions for the shadow prices are as follows:

$$\lambda_1^* = \frac{\sqrt{T_0}}{\mu_P - \mu_B}\left[r(\boldsymbol{\omega},\theta)(\mu_P - \mu_C) - d\right] \tag{8}$$

$$\lambda_2^* = \frac{r(\boldsymbol{\omega},\theta) - b}{a} \tag{9}$$

By combining equations (7), (8) and (9), for a given level of $T_0$ we get

$$\begin{aligned} \boldsymbol{\omega}^* &= \frac{\mu_P - \mu_B}{\mathrm{D}(\boldsymbol{\omega},\theta)} \left\{ \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \left[\frac{r(\boldsymbol{\omega},\theta)}{a} - \mu_C\right]\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota} - \frac{1}{\mu_P - \mu_B}\left[r(\boldsymbol{\omega},\theta)(\mu_P - \mu_C) - d\right]\boldsymbol{\omega_B} \right\} \\ &= -\frac{1}{\mathrm{D}(\boldsymbol{\omega},\theta)}\left[r(\boldsymbol{\omega},\theta)(\mu_P - \mu_C) - d\right]\boldsymbol{\omega_B} + b\frac{\mu_P - \mu_B}{\mathrm{D}(\boldsymbol{\omega},\theta)}\boldsymbol{\omega_Q} + a\frac{\mu_P - \mu_B}{\mathrm{D}(\boldsymbol{\omega},\theta)}\left[\frac{r(\boldsymbol{\omega},\theta)}{a} - \mu_C\right]\boldsymbol{\omega_C}, \end{aligned}$$

where $\boldsymbol{\omega}^*$ is the optimal portfolio for a given value of the tracking error volatility $T_0$,[7] $\mathrm{D}(\boldsymbol{\omega},\theta) \equiv d - \Delta_1 r(\boldsymbol{\omega},\theta)$, while $\boldsymbol{\omega_Q} = b^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and $\boldsymbol{\omega_C} = a^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}$ are the 'Maximum Sharpe-Ratio' and the 'Global Minimum Variance' portfolios lying on the MVF.

Therefore, an implicit definition of the optimal portfolio $\boldsymbol{\omega}^*$ can be given as

$$\boldsymbol{\omega}^* = x_1(\boldsymbol{\omega}^*)\,\boldsymbol{\omega_B} + x_2(\boldsymbol{\omega}^*)\,\boldsymbol{\omega_Q} + x_3(\boldsymbol{\omega}^*)\,\boldsymbol{\omega_C}, \tag{10}$$

where

$$\begin{cases} x_1(\boldsymbol{\omega}^*) = 1 - \dfrac{r(\boldsymbol{\omega}^*,\theta)}{\mathrm{D}(\boldsymbol{\omega}^*,\theta)}(\boldsymbol{\omega}^* - \boldsymbol{\omega_B})'\boldsymbol{\mu} \\ x_2(\boldsymbol{\omega}^*) = \dfrac{b}{\mathrm{D}(\boldsymbol{\omega}^*,\theta)}(\boldsymbol{\omega}^* - \boldsymbol{\omega_B})'\boldsymbol{\mu} \\ x_3(\boldsymbol{\omega}^*) = \left[\dfrac{r(\boldsymbol{\omega}^*,\theta) - b}{\mathrm{D}(\boldsymbol{\omega}^*,\theta)}\right](\boldsymbol{\omega}^* - \boldsymbol{\omega_B})'\boldsymbol{\mu}, \end{cases} \tag{11}$$

and our Risk Balancing Frontier can be thought of as the set of the points on the $(\sigma_P, \mu_P)$ space corresponding to the portfolios $\boldsymbol{\omega}^*$ for any given level of $T_0$. As equation (10) shows, these portfolios can be represented as a linear combination of three well-known portfolios, namely the benchmark portfolio, the 'Maximum Sharpe-Ratio' portfolio, and the 'Global Minimum Variance' portfolio, with the three scalar

[7] In order to avoid excessively burdensome notation, we use the notation $\boldsymbol{\omega}^*$ instead of $\boldsymbol{\omega}^*(T_0)$.

weights summing to unity; these weights are functions of $\boldsymbol{\omega}^*$ and may be outside the $[0,1]$ interval. Since the benchmark belongs to the RBF by definition when $T_0 = 0$, we get $x_1(\boldsymbol{\omega}^*) = 1$, $x_2(\boldsymbol{\omega}^*) = 0$ and $x_3(\boldsymbol{\omega}^*) = 0$; conversely, we do not get the triples $\boldsymbol{x}(\boldsymbol{\omega}^*) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}'$ and $\boldsymbol{x}(\boldsymbol{\omega}^*) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$ for portfolios $\boldsymbol{\omega_Q}$ and $\boldsymbol{\omega_C}$ because they do not lie on the RBF.

The triples with $x_1(\boldsymbol{\omega}^*) = 0$, $x_2(\boldsymbol{\omega}^*) \neq 0$ and $x_3(\boldsymbol{\omega}^*) \neq 0$ deserve special attention because equation (10) reduces to the well-known Mutual Fund Separation Theorem (Merton, 1972) in which any portfolio belonging to the Mean-Variance boundary can be written as a proper linear combination of $\boldsymbol{\omega_C}$ and $\boldsymbol{\omega_Q}$. Under this condition the RBF and the efficient branch of the MVF must have a common portfolio.

Note that an explicit solution to equation (10) cannot be found analytically, and numerical techniques are called for. A convenient method to determine the *locus* is to apply the BFGS numerical optimization algorithm for a grid of values for $T_0$.[8]

A description of the algorithm is best given by referring to the geometrical objects depicted in Figure 2. For any given level of the restriction $T_0$, the return of the tangency portfolio $K$ will lie in the interval $\mu_0(T_0) \leq \mu_K \leq \mu_1(T_0)$, where $\mu_0(T_0)$ and $\mu_1(T_0)$ are the TEV-dependent expected returns of the endpoints $J_0$ and $J_1$. These are the CTF-constrained minimum variance allocation

$$J_0 \equiv (\sigma_B^2 + T_0 - 2\sqrt{T_0 \Delta_2}, \mu_B - \Delta_1 \sqrt{T_0/\Delta_2}) \tag{12}$$

and the one with the highest expected return

$$J_1 \equiv (\sigma_B^2 + T_0 + 2\Delta_1 \sqrt{T_0/d}, \mu_B + \sqrt{dT_0}). \tag{13}$$

Point $J_1$ also corresponds to the position where the hyperbolic MTF crosses the oval CTF (see Jorion, 2003; Palomba and Riccetti, 2019). Consequently, for any given expected return $\mu \in [J_0(T_0), J_1(T_0)]$, the function

$$S(T_0, \mu) = \sigma_B^2 + T_0 + \frac{2}{d} \left\{ \Delta_1 (\mu - \mu_B) - \sqrt{d \delta_B [dT_0 - (\mu - \mu_B)^2]} \right\} \tag{14}$$

returns the risk for the portfolio on the arc $\widehat{J_0 J_1}$ which minimizes the VaR for a given $T_0$.

The RBF is found by calculating (14) on a numerical grid $T_0 = 0, h, 2h, 3h, \ldots, T^{\max}$, where $h$ is an arbitrary and numerically small increment. Our algorithm can therefore be described as:

1. starting from $T_0 = 0$, calculate the extremal returns $\mu_0(T_0)$ and $\mu_1(T_0)$,

2. set $\bar{\mu}$ as the midpoint between $\mu_0(T_0)$ and $\mu_1(T_0)$,

[8] See Broyden (1970); Fletcher (1970); Goldfarb (1970); Shanno (1970).

3. minimize numerically the VaR $V(\mu) = z_\theta \sqrt{S(T_0, \mu)} - \mu$ via the BFGS algorithm using $\bar{\mu}$ as a starting point and call $\mu^*$ the solution;

4. determine the coordinates of the resulting portfolio $\boldsymbol{\omega}^*$ using equation (14),

5. increment $T_0$ by $h$ and repeat until $T_0 = T^{\max}$.

For each point on the RBF, the corresponding portfolio $\boldsymbol{\omega}^*$ can be found using the method outlined in Appendix B.

## 3.3 Geometrical properties of the RBF in the standard case

The RBF, as defined in the previous section, is a continuum of portfolios, indexed by the TEV $T_0$; when $T_0 = 0$, $\boldsymbol{\omega}^*$ is the benchmark portfolio, and different choices for $T_0$ lead to different optima $\boldsymbol{\omega}^*$. This set can be split into the following 3 non-overlapping subsets for increasing values of $T_0$:

$$\text{RBF} = \text{RBF}_1 \,\cup\, \text{RBF}_2 \,\cup\, \text{RBF}_3.$$

These subsets are identified by the fact that

1. there exists a TEV level $T_Z$ such that the variance of the portfolio $\boldsymbol{\omega_Z} = \boldsymbol{\omega}(T_Z)$ is a minimum within the RBF (see Appendix A for a proof);

2. there exists a TEV level $T_M$ such that portfolio $\boldsymbol{\omega_M} = \boldsymbol{\omega}(T_M)$ minimizes the VaR. Since we assume that the manager's confidence level $z_\theta$ is high, such portfolio is Markowitz-efficient. From a geometrical point of view, the minimum VaR portfolio $M$ is the tangency portfolio between the linear CVF and the Markowitz' MVF. The analytical proof is provided in section 3.3.2.

It is useful to distinguish the two cases $T_Z \leq T_M$, which we refer to as the "standard" case, and the reverse case $T_Z > T_M$, that occurs when a benchmark with high risk and return is chosen. We call this case the "aggressive benchmark" case. In the former case we have

$$\begin{aligned} &\text{RBF}_1 = \{\boldsymbol{\omega}^* \colon 0 \leq \text{TEV} \leq T_Z\}, \quad \text{RBF}_2 = \{\boldsymbol{\omega}^* \colon T_Z < \text{TEV} \leq T_M\}, \\ &\text{RBF}_3 = \{\boldsymbol{\omega}^* \colon \text{TEV} > T_M\} \end{aligned} \tag{15}$$

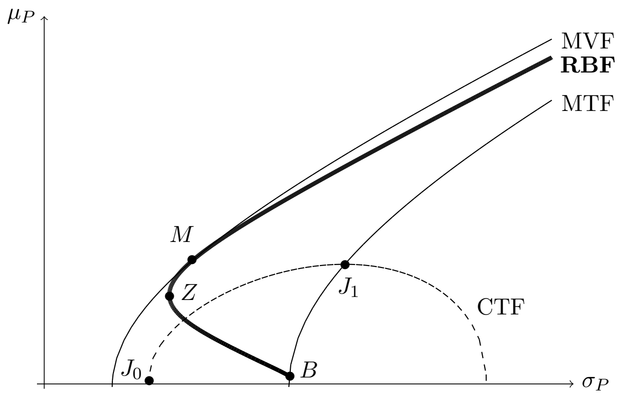
while, in the aggressive case, these subsets are defined as

$$\begin{aligned} &\text{RBF}_1 = \{\boldsymbol{\omega}^* \colon 0 \leq \text{TEV} < T_M\}, \quad \text{RBF}_2 = \{\boldsymbol{\omega}^* \colon T_M \leq \text{TEV} \leq T_Z\}, \\ &\text{RBF}_3 = \varnothing. \end{aligned} \tag{16}$$

These subsets possess different properties from the financial point of view. We will focus on the standard case first and leave the analysis of the aggressive case for subsection 3.4.

As claimed in Section 3.1, the RBF is an envelope that contains all the tangency portfolios between the CTF and CVF curves. Analytically, each point of this frontier yields the solution of a system containing both equations (1) and (2). Palomba and Riccetti (2012) show that the solution is a fourth degree equation in the mean-variance space and proved that the solution depends on the benchmark coordinates together with the values of $T_0$, $V_0$ and the confidence level $\theta$.

Figure 3: The RBF in the standard case




From Figure 3 several important characteristics of the RBF become apparent: the boundary on the $(\sigma_P, \mu_P)$ space takes a horseshoe-like shape, with one endpoint necessarily at $B$, where $T_0 = 0$. The subsets $\text{RBF}_1$ and $\text{RBF}_2$ correspond to the arcs $\widehat{BZ}$, $\widehat{ZM}$, and the points to the right of $M$ form the subset $\text{RBF}_3$. The situation in which the benchmark portfolio lies on the efficient branch[9] of the MVF is notable, since in this case the arc $\widehat{BM}$ on the RBF lies on the MVF.

### 3.3.1 The arc $\widehat{\mathbf{BZ}}$

Since portfolio $B$ identifies the passive strategy $T_0 = 0$, as the TEV increases the arc $\widehat{J_0 J_1}$ moves away from $B$; therefore, as a rule, the arc $\widehat{BZ}$ can be thought of a line starting from $B$ and going in the North-West direction, with decreasing efficiency loss. In practice, for each portfolio $P \in \text{RBF}_1$, asset managers obtain $\sigma_P \leq \sigma_B$ and $V_P \leq V_B$.

Let $Z \equiv (\sigma_Z, \mu_Z)$ be the minimum variance portfolio lying on the RBF: the existence of such portfolio is proven in Appendix A and implies that asset managers

[9]The situation of a benchmark located in the inefficient branch would have no practical relevance.

can jointly satisfy the pair of constraints TEV = $T_0$ and VaR = $V_0$ so as to select a position in which they can also minimize overall portfolio risk. Thus, the arc $\widehat{BZ}$ is the subset of the RBF where an increase in the TEV limit implies a tighter VaR restriction, leading to more efficient portfolios. In other words, along the $\widehat{BZ}$ arc the TEV and VaR move in opposite directions, and a trade-off exists between relative risk (the TEV) and both measures of absolute portfolio risk (the VaR and $\sigma_P$).

### 3.3.2 The arc $\widehat{\mathbf{ZM}}$

In order to analyze the properties of the intermediate subset $\text{RBF}_2$, we begin by considering the portfolio

$$\hat{\boldsymbol{\omega}} = \operatorname*{argmin}_{\boldsymbol{\omega}\in\text{RBF}} VaR(\boldsymbol{\omega});$$

that minimizes the VaR along the RBF for a given TEV = $T_0$. Since the first shadow price in equation (8) yields the variation of the objective function with respect to $T_0$

$$\frac{\partial VaR(\boldsymbol{\omega})}{\partial T_0} = \sqrt{T_0}\frac{r(\boldsymbol{\omega},\theta)(\mu_P-\mu_C)-d}{\mu_P-\mu_B},$$

then $\hat{\boldsymbol{\omega}}$ must satisfy

$$\frac{\partial VaR(\boldsymbol{\omega})}{\partial T_0} = 0 \qquad \Rightarrow \qquad \hat{\mu} = \hat{\boldsymbol{\omega}}'\boldsymbol{\mu} = \mu_C + \frac{d}{z_\theta}\sigma(\hat{\boldsymbol{\omega}}). \tag{17}$$

Now consider $M$, defined earlier as the portfolio that minimizes the VaR among admissible portfolios: Palomba and Riccetti (2012) prove that its coordinates on the $(\sigma_P, \mu_P)$ space are

$$M \equiv \left(\frac{z_\theta^2}{z_\theta^2-d}\sigma_C^2, \mu_C + d\frac{\sigma_C}{\sqrt{z_\theta^2-d}}\right). \tag{18}$$

Geometrically, $M \equiv (\sigma_M, \mu_M)$ is the contact portfolio between the MVF and the linear boundary CVF, and the associated VaR equals $V_M = z_\theta\sigma_M - \mu_M = -\mu_C + \sqrt{\sigma_C^2(z_\theta^2-d)}$. This is the most binding assignable VaR because lower values lead to infeasible portfolios.

Since

$$\mu_M = \mu_C + \frac{d}{z_\theta}\sigma(\hat{\boldsymbol{\omega}}) = \mu_C + \frac{d\sigma_C}{\sqrt{z^2-d}} = \hat{\mu}, \tag{19}$$

portfolio $M$ satisfies condition (17). From this result it is easy to see why the RBF takes the shape shown in Figure 2: starting from the benchmark $B$, we have $(\boldsymbol{\omega_P}-\boldsymbol{\omega_B})'\boldsymbol{\mu} \geq 0$, and therefore the VaR decreases first until it reaches its minimum at portfolio $M$, and then it rises again for $\mu_P > \mu_M$.

Most of the properties of the points along $\widehat{ZM}$ are the same as those of $\widehat{BZ}$; notably, moving from $Z$ to $M$, and thus allowing for larger TEV, still leads to portfolios that have lower VaR and feature decreasing efficiency loss. However, contrary to the $\widehat{BZ}$ case, this comes at the price of a larger variance.

### 3.3.3 The upper branch

This subset of the RBF must be considered because the optimization problem (3) is defined with an equality constraint, rather than an inequality, as would perhaps be more appropriate. Therefore, we discuss it here for the sake of completeness, but its practical relevance is more limited than the previous two subsets.

By a simple economic argument, under an inequality constraint TEV $\leq T_0$ the RBF stops at $M$: from point $M$ onward, there are progressively less stringent VaR constraints and the portfolio risk grows larger. The upper branch of the RBF defines an increasing line in the $(\sigma_P, \mu_P)$ space that lies underneath the MVF and moves away from it in the North-East direction. This part of the boundary is the least attractive to investors with a high level of risk aversion: despite the portfolio returns being large, increasing the tolerable risk relative to the benchmark $B$ leads to riskier portfolios in terms of overall variance and Value-at-Risk. Hence, the point $M$ can be thought of as a watershed between two opposite TEV-VaR relationships. Given these premises, we may call this subset the "daredevil" segment of the RBF.

Figure 4: Relationships along the RBF

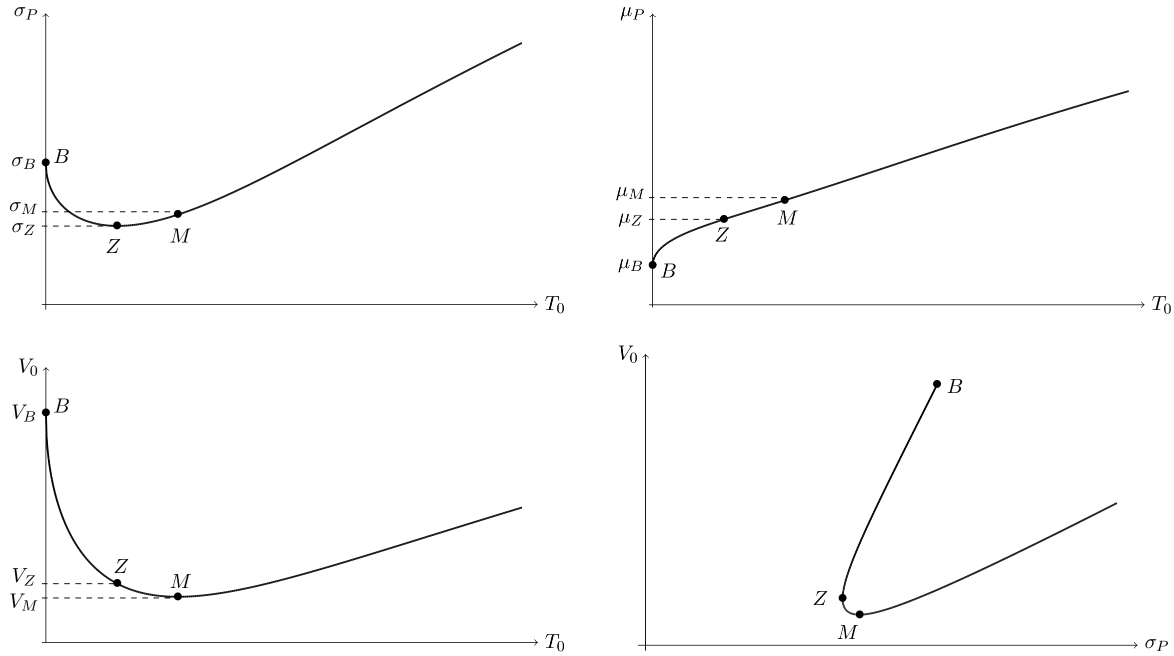


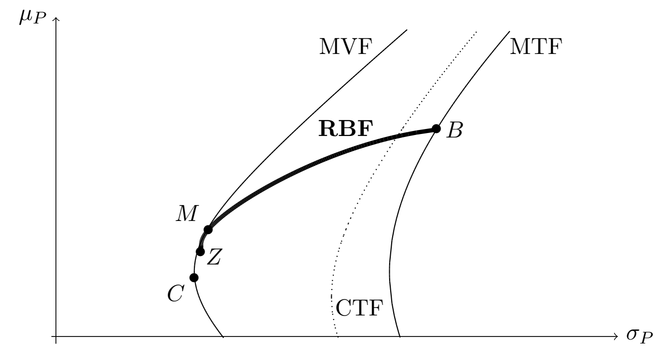

Figure 4 summarizes the relationships along the RBF between all the relevant portfolio quantities; note the special importance of the three reference portfolios $B$, $Z$ and $M$. Moving from the benchmark $B$, the portfolio variance and the VaR follow a convex and non-monotonic relationship, while the expected return of the portfolios always increases. The lower-right plot displays the relationship between the two absolute risk measures. Starting from the benchmark, the overall initial risk reduction is accompanied by a progressively more stringent VaR. From portfolio $Z$ onward, the overall portfolio variance increases, while the efficiency loss reduces until the portfolio $M$ is reached. From the point $M$ onward the efficiency loss increases and is accompanied by progressively larger values of $T_0$ and $V_0$.

The discussion above entails an important implication: setting a maximum value of the TEV lower than the one of the $Z$ portfolio is a questionable choice for the risk manager, since along the $\widehat{BZ}$ arc either the expected return and all the absolute risk indicators can be improved by raising the TEV. Moving along the upper branch, any choice beyond $M$ would only lead to extremely risky positions, so it would be a rather extreme one in practice.

## 3.4 The aggressive benchmark

In the "aggressive benchmark" case, the benchmark $B$ is itself a high risk-high return portfolio. This situation arises when the benchmark return and variance are greater than the one of the minimum VaR portfolio $M$. In this context, $B$ is the portfolio with the highest available expected return on the RBF, but this feature is linked to a strong risk. As a consequence, loosening the TEV constraint makes it optimal to reduce the associated risk, rather than increase the portfolio return, and equation (16) holds instead of (15). Figure 5 shows the shape of the RBF in the case of an aggressive benchmark.

Figure 5: Aggressive benchmark




The boundary starts from the benchmark and proceeds South-West, reaches the tangency portfolio $M$ and stops at the point of minimum risk $Z$.

Technically, the RBF can be extended beyond portfolio $Z$ for values of the TEV higher than $T_Z$, but from a financial point of view this would lead to inefficient portfolios, *i.e.* dominated under any possible metric (variance, return, VaR or portfolio efficiency loss).[10] Therefore, we truncate the boundary at $Z$ and set the condition $\text{RBF}_3 = \varnothing$ in equation (16). Using equation (17), the condition $(\boldsymbol{\omega_P} - \boldsymbol{\omega_B})'\boldsymbol{\mu} < 0$ always holds so the VaR increases all along the arc $\widehat{ZM}$.

### 3.5 Economic and financial implications

In this section, we analyze the properties of the RBF for active portfolio management from the viewpoints of the asset manager and of the investor. As argued above, VaR and TEV constraints are such that the RBF is non-empty and a VaR-TEV efficient choice is possible; the question then is: how should the two parties choose the portfolio among the Pareto-optimal ones?

In order to address this question, we consider the misalignment of investors' objectives with those of asset managers. Financial literature points out that delegated asset management involves layering of agency relationships, and the number of conflicts of interest between investors and their agents is likely to increase together with the complexity of agency relationships, with consequences on the optimal compensation contract. The incentive issue is very important, as portfolio theory field customarily builds on the idea that the agent could be led to selecting higher-risk and higher-return assets out of the structure of contract compensation. Theoretically, aligning the objectives of the investor and of the asset manager implies a simultaneous control on returns through the profit sharing rule, on performance measures against a benchmark and on risk-taking through the constraints on the TEV (Bank for International Settlements, 2003).

Our work is based on the investor's customary choice to set limits to TEV and VaR so as to limit risk exposure in a context when the asset manager's compensation depends on the ability to outperform the benchmark in terms of return ($\mu_P > \mu_B$). Thus, the asset manager could face sanctions if the VaR or, more importantly, the TEV exceeds the maximum threshold.

Given this limitation, the asset manager's optimal strategy might be to keep the TEV low and stick close to the benchmark, while placing a few "bets" by over-weighting assets that are likely to over-perform. Thus, it is also possible to keep the VaR close to the benchmark's, with relatively little effort on the manager's side. This behavior has been described by several authors, such as Ineichen (2004), who focuses on the poor returns of the relative performance paradigm. Indeed, this

[10] In principle, this may not be strictly true, as it is conceivable that one could reach portfolios with a higher return than $\mu_B$ for very large values of the TEV. However, we consider this scenario as extremely unrealistic.

framework can encourage asset managers to take too much risk or to be too passive. On one hand, Roll (1992) already noted that excess return optimization leads to portfolios with a systematically higher risk than the benchmark; other authors, such as Scowcroft and Sefton (2001), questioned the utility of a constraint on the TEV. Jorion (2003) notes that the excess return-TEV framework induces the asset manager to optimize in an excess return space only and to ignore the investor's overall portfolio risk. However, the asset management industry maintains an emphasis on relative risk, since absolute risk may simply be a consequence of the volatility of the benchmark. Therefore, Jorion (2003) attempts to correct the problem of portfolios with higher variance than the benchmark by placing an additional constraint (beyond the maximum TEV) that forces portfolio volatility to be equal to that of the benchmark.

On the other hand, the investor's objective function depends positively on the portfolio return and negatively on its absolute risk, as measured either by the VaR or by its variance. If the benchmark performs poorly, the investor should benefit from a strongly active management that reduces the losses even by increasing the TEV. Riccetti (2012) shows that risk managers should bind the asset manager to a minimum TEV level, so as to ensure that the asset manager is effectively undertaking an active strategy; Lo (2008) contains an analysis on the problem of measuring how active the asset manager is.

The inverse relationship between TEV and VaR along the Risk Balancing Frontier produces a trade-off between the different objectives of the two parties. From this perspective, the RBF could be seen as a useful tool in analyzing the misalignment in objectives. For instance, Admati and Pfleiderer (1997) argue that the rationale for a benchmark-adjusted compensation scheme is inconsistent with optimal risk-sharing and does not help in solving potential contracting problems with the manager. In other words, if the investor is actively involved in the investment decisions the resulting portfolio could be close to the Mean-Variance frontier, or possibly on it. Otherwise, the main role is played by the asset manager, with two possible scenarios. In one case, the constraint on the TEV can be very strict, and the asset manager has little choice but to stick close to the benchmark and make very little effort. Conversely, if the constraint on the TEV is too loose, then the asset manager may end up selecting an overly risky portfolio in order to maximize its return. Other contributors (He and Xiong, 2013) find that tight tracking error constraints, alongside narrow mandates, can provide incentives for agents.

By focusing on the RBF, all the situations above can be clearly detected. The optimal position for the investor is associated with portfolios near $M$, while the other two cases are associated with portfolios close to $B$ and any portfolio lying on the upper branch. The investor should steer the asset manager towards an active strategy, while keeping the portfolio risk under control at the same time: this translates into a maximum TEV greater than or equal to the one of portfolio $M$, and the setting of suitable maxima for the VaR and/or variance constraints.

# 4 Empirical example

In order to show how our RBF boundary works in practice, we carry out a short numerical example, using the S&P100 market index as a benchmark and all its constituents, listed in Table C.1 in Appendix C, as the universe of available risky securities. The daily returns for both index and stocks were calculated using one year of data covering two distinct periods: a pre-COVID period ranging from 2019-01-01 to 2019-12-31, and the period that goes from 2020-04-01 to 2021-03-31, which we define as the "post-COVID period". The use of both periods is necessary to make a comparison because, as illustrated by Figure 6, returns and volatility are generally higher during the "post-COVID" period. With this brief empirical exercise, we illustrate the behavior of the RBF in the $(\sigma_P, \mu_P)$ space when the socio-economic environmental conditions undergo a dramatic change, such as the one induced by the COVID pandemic on the US stock market.

Figure 6: Constituents of S&P100 index in the two subsamples

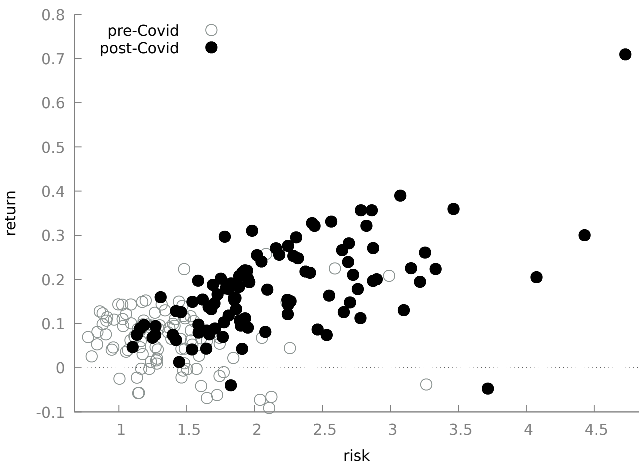



Table 1 reports several statistics for some portfolios of interest, namely the benchmark, the minimum variance and the minimum VaR on the RBF, and the Global Minimum Variance and the Maximum Sharpe Ratio on the MVF.

Table 1: Portfolio results

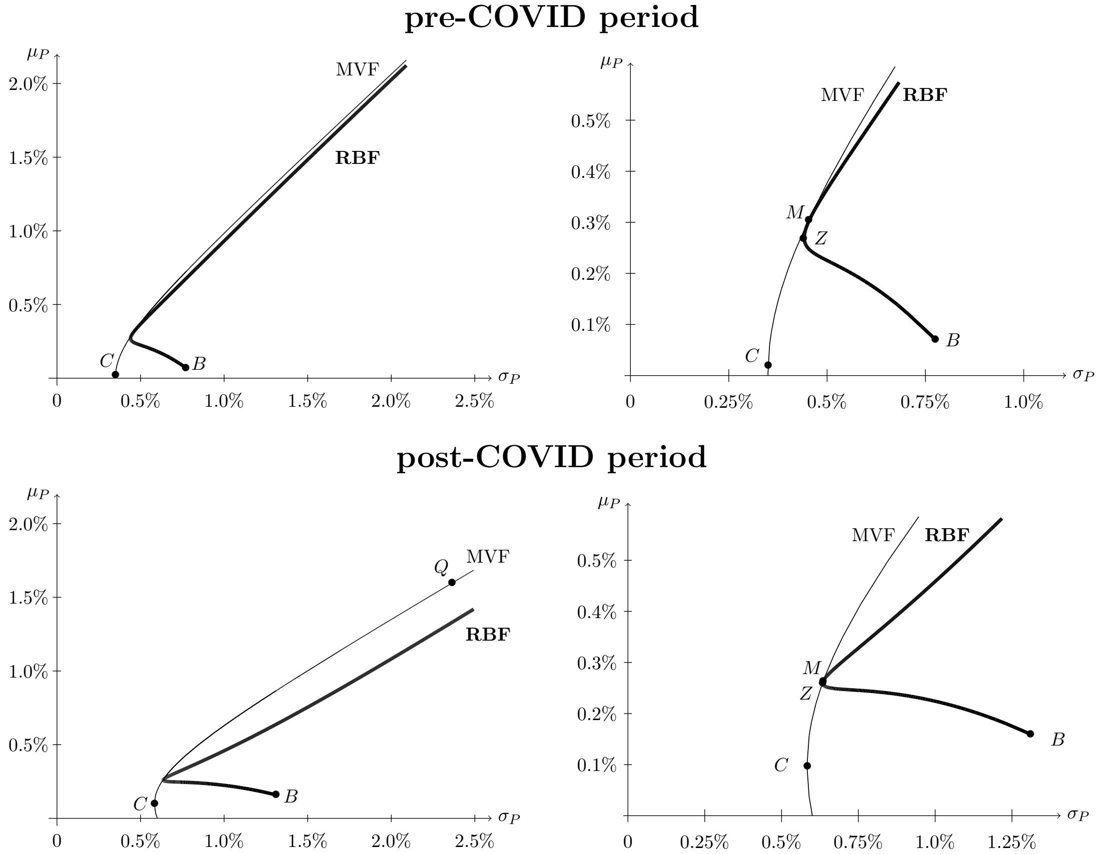
| | | | pre-COVID period | | | | | post-COVID period | | |
|---|---|---|---|---|---|---|---|---|---|---|
| - MVF/MTF asymptotic slope ($\sqrt{d}$) | | | | | 1.0460 | | | | | 0.6547 |
| - confidence level at $\sqrt{d}$ | | | | | 0.8523 | | | | | 0.7437 |
| - $\Delta_1 = \mu_B - \mu_C$ | | | | | 0.0665 | | | | | 0.0627 |
| - $\Delta_2 = \sigma_B^2 - \sigma_C^2$ | | | | | 0.4787 | | | | | 1.3770 |
| **Portfolios:** | $B$ | $C$ | $Q$ | $Z$ | $M$ | $B$ | $C$ | $Q$ | $Z$ | $M$ |
| Return | 0.0696 | 0.0031 | 43.3041 | 0.2696 | 0.3046 | 0.1598 | 0.0970 | 1.5964 | 0.2594 | 0.2625 |
| Risk (st. dev) | 0.7752 | 0.3496 | 41.3870 | 0.4398 | 0.4530 | 1.3102 | 0.5827 | 2.3632 | 0.6341 | 0.6351 |
| Sharpe ratio | 0.0898 | 0.0088 | 1.0463 | 0.6129 | 0.6724 | 0.1219 | 0.1666 | 0.6755 | 0.4090 | 0.4134 |
| Tracking Error | 0.0000 | -0.0665 | 43.2345 | 0.2000 | 0.2350 | 0.0000 | -0.0627 | 1.4367 | 0.0996 | 0.1028 |
| TEV | 0.0000 | 0.4787 | 1707.9782 | 0.4074 | 0.5251 | 0.0000 | 1.3771 | 6.1836 | 1.3146 | 1.3925 |
| Information Ratio* | - | -0.1390 | 0.0253 | 0.4908 | 0.4475 | - | -0.0455 | 0.2323 | 0.0758 | 0.0738 |
| Efficiency loss | 0.4747 | 0.0000 | 0.0000 | 0.0064 | 0.0000 | 1.3679 | 0.0000 | 0.0000 | 0.0011 | 0.0000 |
| VaR | 1.2054 | 0.5719 | 24.7715 | 0.4539 | 0.4406 | 1.9953 | 0.8614 | 2.2908 | 0.7836 | 0.7822 |
| $x_1(\boldsymbol{\omega})$ | 1.0000 | - | - | 0.1161 | 0.0000 | 1.0000 | - | - | 0.0283 | 0.0000 |
| $x_2(\boldsymbol{\omega})$ | 0.0000 | - | - | 0.0060 | 0.0070 | 0.0000 | - | - | 0.1071 | 0.1104 |
| $x_3(\boldsymbol{\omega})$ | 0.0000 | - | - | 0.8779 | 0.9930 | 0.0000 | - | - | 0.8646 | 0.8896 |

**Note:** $\theta = 0.95$ (High confidence level in both periods), CVF slope $z_\theta = 1.645$;
Risk Balancing Frontier (RBF): maximum TEV $T^{\max} = 8.0$, TEV increments: $h = 10^{-4}$.
* Information Ratio = Tracking Error/TEV (see *e.g* Lee, 2000)

Figure 7 illustrates the MVF and the RBF boundaries for both periods. The plots on the left offer a global view; those on the right zoom in around the RBF minimum variance portfolio $Z$. As can be noticed, the risk-return coordinates of the benchmark portfolio are very different between the two periods; during the pandemic phase, the horizontal distance between $B$ and the MVF is much greater, so the TEV cannot be kept under control without serious efficiency losses. The graph also shows that during the COVID period the portfolios have become riskier and with higher expected returns. The only exception is the 'Maximum Sharpe Ratio' portfolio $Q$ whose values were also quite extreme, in terms of both risk and return, before the pandemic. It is interesting to note that the efficiency loss for the $Z$ portfolio decreases in the COVID period, despite the drop in the portfolio return and the risk increase. It should also be noted that portfolio $Z$ displays a much higher TEV in the COVID period and that this may have led managers to reduce the VaR.

The shift of the RBF towards riskier positions is mirrored by the adjustment of the $x_i(\boldsymbol{\omega})$ weights for the $Z$ and $M$ portfolios, and especially by the increase in $x_2(\boldsymbol{\omega})$ (see equation (10)). Therefore, there is a shift in favor of portfolio $Q$, whose TEV is much higher in the pre-COVID period. On the other hand, the shift of the MVF and RBF due to the instability caused by the pandemic has increased the distance between the upper branches of the two curves: the efficiency loss for an aggressive portfolio, which is minimal in the pre-COVID period, becomes quite substantial afterwards.

Figure 7: The Markowitz and Risk Balancing frontiers: before and after




# 5 Concluding remarks

In this paper, we develop a novel tool to analyze the issue of misalignment in objectives of investor and asset manager in the case of actively managed portfolios. In doing so, we consider those situations when asset managers must jointly satisfy the restrictions imposed by investors on two risk measures: the tracking error volatility (TEV) and the Value-at-Risk (VaR). This framework creates a trade-off possibility between the two constraints: when the TEV and the VaR restrictions hold at the same time, the more stringent is one, the less binding is the other. The tool we propose to analyze this situation is the Risk Balancing Frontier (RBF), a portfolio boundary in the risk-return space that identifies all portfolios with minimum VaR given a preset TEV level. We prove that the RBF can be expressed as a combination of three basic portfolios, namely the benchmark, the 'Maximum Sharpe Ratio' and the 'Global Minimun Variance' portfolios; we also study the boundary's main geometrical properties, and provide operational details about the computational issues involved.

From the financial point of view, our contribution proves that the RBF sheds light on the different relationships between the two risk measures by considering the RBF either as a whole or via its three separate subsets, with different characteristics, described in section 3.3. In our opinion, this boundary could help solve the traditional agency problem specific to delegated portfolio management. In order to exemplify the practical usage of the RBF for analyzing actual market scenarios, we provide an analysis of the SP100 index before and after the COVID pandemic.

Our approach could be further developed in several directions. For example, introducing a risk free asset, management fees or transaction costs would certainly make the analysis more realistic. Another possibility is to disallow short selling. This corresponds to a severe restriction that often characterizes different fund policies or contracts between managers and investors, but it comes at the cost of making the algebra and the computational aspects much more complex. Finally, removing the hypothesis of normally distributed returns would certainly represent a natural extension: this possibility would enrich our analysis by generalizing for non-standard distributions of returns but, on the other hand, would require a substantial and complex revision of all portfolio frontiers involved in our proposed approach.

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# Appendix A: The portfolio variance along the RBF

Equation (10) can be re-expressed as

$$\boldsymbol{\omega}^* = \boldsymbol{W}\boldsymbol{x}(\boldsymbol{\omega}^*)$$

where $\boldsymbol{W} = \begin{bmatrix}\boldsymbol{\omega_B} & \boldsymbol{\omega_Q} & \boldsymbol{\omega_C}\end{bmatrix}$ is a $n\times 3$ matrix and $\boldsymbol{x}(\boldsymbol{\omega}^*)' = \begin{bmatrix}x_1(\boldsymbol{\omega}^*) & x_2(\boldsymbol{\omega}^*) & x_3(\boldsymbol{\omega}^*)\end{bmatrix}$.

The variance of each portfolio lying on the RBF is

$$\sigma^{*2} = \boldsymbol{\omega}^{*\prime}\boldsymbol{\Sigma}\boldsymbol{\omega}^* = \boldsymbol{x}(\boldsymbol{\omega}^*)'\boldsymbol{\Omega}\boldsymbol{x}(\boldsymbol{\omega}^*) \tag{A.1}$$

where the matrix $\boldsymbol{\Omega}$ can be obtained as the quadratic form

$$\boldsymbol{\Omega} = \boldsymbol{W}'\boldsymbol{\Sigma}\boldsymbol{W} = \begin{bmatrix}\sigma_B^2 & \mu_B/b & \sigma_C^2\\ \mu_B/b & \sigma_Q^2 & \sigma_C^2\\ \sigma_C^2 & \sigma_C^2 & \sigma_C^2\end{bmatrix} = \frac{2}{b}\begin{bmatrix}b\sigma_B^2 & \mu_B & \mu_C\\ \mu_B & \mu_Q & \mu_C\\ \mu_C & \mu_C & \mu_C\end{bmatrix}.$$

In order to find the variance-minimizing portfolio along the RBF we need to solve the following problem

$$\begin{array}{ll}\min & \sigma^{*2} = \boldsymbol{x}(\boldsymbol{\omega})'\boldsymbol{\Omega}\boldsymbol{x}(\boldsymbol{\omega})\\ \text{sub} & G[\boldsymbol{x}(\boldsymbol{\omega})] = 0,\end{array} \tag{A.2}$$

where

$$G[\boldsymbol{x}(\boldsymbol{\omega})] = \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}-r(\boldsymbol{\omega},\theta)\\ b\\ r(\boldsymbol{\omega},\theta)-b\end{bmatrix}\frac{(\boldsymbol{\omega}-\boldsymbol{\omega_B})'\boldsymbol{\mu}}{\mathrm{D}(\boldsymbol{\omega},\theta)} - \begin{bmatrix}x_1(\boldsymbol{\omega})\\ x_2(\boldsymbol{\omega})\\ x_3(\boldsymbol{\omega})\end{bmatrix}.$$

The problem (A.2) consists of minimizing the portfolio variance in a trivariate system where the portfolio weights are restricted to be those obtained in equation (11). In this context, the restriction is nonlinear but it always guaranteed that such weights sum up to unity. The Lagrangian is

$$L[\boldsymbol{x}(\boldsymbol{\omega}),\theta] = \boldsymbol{x}(\boldsymbol{\omega})'\boldsymbol{\Omega}\boldsymbol{x}(\boldsymbol{\omega}) - \boldsymbol{\lambda}'G[\boldsymbol{x}(\boldsymbol{\omega})],$$

therefore

$$\left.\frac{\partial L[\boldsymbol{x}(\boldsymbol{\omega}),\theta]}{\partial \boldsymbol{x}(\boldsymbol{\omega})}\right|_{\boldsymbol{\omega}=\hat{\boldsymbol{\omega}}} = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{\Omega}\boldsymbol{x}(\hat{\boldsymbol{\omega}}) - \boldsymbol{G}'[\boldsymbol{x}(\hat{\boldsymbol{\omega}})]\boldsymbol{\lambda} = \boldsymbol{0}, \tag{A.3}$$

where $\boldsymbol{G}'[\boldsymbol{x}(\hat{\boldsymbol{\omega}})]$ is the $3\times 3$ Jacobian matrix. Since we have

$$\frac{\partial r(\boldsymbol{\omega},\theta)}{\partial \boldsymbol{x}(\boldsymbol{\omega})} = -\frac{r(\boldsymbol{\omega},\theta)}{\sigma^2}\boldsymbol{\Omega}\boldsymbol{x}(\boldsymbol{\omega}), \quad \frac{\partial \mathrm{D}(\boldsymbol{\omega},\theta)}{\partial \boldsymbol{x}(\boldsymbol{\omega})} = \Delta_1\frac{r(\boldsymbol{\omega},\theta)}{\sigma^2}\boldsymbol{\Omega}\boldsymbol{x}(\boldsymbol{\omega}) \quad \text{and} \quad \frac{\partial(\boldsymbol{\omega}-\boldsymbol{\omega_B})'\boldsymbol{\mu}}{\partial \boldsymbol{x}(\boldsymbol{\omega})} = \boldsymbol{W}'\boldsymbol{\mu},$$

where $\boldsymbol{\omega} = \boldsymbol{W}\boldsymbol{x}(\boldsymbol{\omega})$ and $\boldsymbol{W}'\boldsymbol{\mu} = \begin{bmatrix}\mu_B & \mu_Q & \mu_C\end{bmatrix}'$, we get

$$\boldsymbol{G}'[\boldsymbol{x}(\boldsymbol{\omega})] \quad = \quad \begin{bmatrix} -\boldsymbol{x}(\boldsymbol{\omega})'\boldsymbol{\Omega}\,\phi[\mathrm{D}(\boldsymbol{\omega},\theta) - \Delta_1 r(\boldsymbol{\omega},\theta)] - \boldsymbol{\mu}'\boldsymbol{W}\dfrac{r(\boldsymbol{\omega},\theta)}{\mathrm{D}(\boldsymbol{\omega},\theta)} \\ -\boldsymbol{x}(\boldsymbol{\omega})'\boldsymbol{\Omega}\,\phi b\Delta_1 + \boldsymbol{\mu}'\boldsymbol{W}\dfrac{b}{\mathrm{D}(\boldsymbol{\omega},\theta)} \\ -\boldsymbol{x}(\boldsymbol{\omega})'\boldsymbol{\Omega}\,\phi(d - b\Delta_1) + \boldsymbol{\mu}'\boldsymbol{W}\dfrac{r(\boldsymbol{\omega},\theta) - b}{\mathrm{D}(\boldsymbol{\omega},\theta)} \end{bmatrix}' - \boldsymbol{I_3}$$

where $\phi = \dfrac{(\mu^* - \mu_B)}{\mathrm{D}(\boldsymbol{\omega},\theta)^2}\dfrac{r(\boldsymbol{\omega},\theta)}{\sigma^2}$ is a scalar and $\boldsymbol{I_3}$ is the $3 \times 3$ identity matrix.

Assuming that the matrix $\boldsymbol{\Omega}$ is non-singular, from equation (A.3) the solution is

$$\boldsymbol{x}(\hat{\boldsymbol{\omega}}) = \boldsymbol{\Omega}^{-1}\boldsymbol{G}'[\boldsymbol{x}(\hat{\boldsymbol{\omega}})]\boldsymbol{\lambda}, \tag{A.4}$$

and therefore, after substituting $\boldsymbol{G}[\boldsymbol{x}(\hat{\boldsymbol{\omega}})]$ into the constraint, we get

$$\hat{\boldsymbol{\lambda}} = \{\boldsymbol{\Omega}^{-1}\boldsymbol{G}'[\boldsymbol{x}(\hat{\boldsymbol{\omega}})]\}^{-1}\boldsymbol{x}(\hat{\boldsymbol{\omega}}) = \boldsymbol{G}'[\boldsymbol{x}(\hat{\boldsymbol{\omega}})]^{-1}\boldsymbol{\Omega}\boldsymbol{x}(\hat{\boldsymbol{\omega}}).$$

Since by assumption $\boldsymbol{\Omega}$ is positive definite, the existence of the solution (A.4) guarantees that the RBF admits a portfolio in which the overall risk is minimized.

# Appendix B: Points and portfolio weights

For each $n \times 1$ vector $\boldsymbol{\omega_P}$ containing the portfolio weights, there exists a corresponding point $P$ on the $(\sigma_P, \mu_P)$ space. Clearly, given $\boldsymbol{\omega_P}$, the $n$-dimensional column vector of expected returns $\boldsymbol{\mu}$ and their covariance matrix $\boldsymbol{\Sigma}$, the coordinates of $P$ can be easily determined by the usual equations

$$\sigma_P = \sqrt{\boldsymbol{\omega_P'}\boldsymbol{\Sigma}\boldsymbol{\omega_P}} \qquad \text{and} \qquad \mu_P = \boldsymbol{\omega_P'}\boldsymbol{\mu}.$$

As for the inverse problem, namely finding the portfolio weights given the couple $[\sigma_P \quad \mu_P]'$, one would have to solve a system of 2 equations in $n$ variables, that obviously has infinitely many solutions. In this Appendix, we describe how to determine the vector of portfolio weights $\boldsymbol{\omega_K}$ corresponding to any point $K$ that lies on the Risk Balancing Frontier.

All portfolios belonging to the RBF share two main characteristics: on the one hand, they are characterized by the minimum VaR that can be reach for each TEV level and, on the other, they are the tangency points between the linear CVF and the "oval" CTF. The minimum VaR value $V_0$ can be calculated via the algorithm we introduced at the end of section 3.2; in this context, information is needed on the benchmark portfolio weights $\boldsymbol{\omega_B}$ and on $T_0$ restriction imposed to the TEV. The tangency condition in the $(\sigma_P, \mu_P)$ space implies that there is a single point $K$ associated to both measures of risk $T_0$ and $V_0$, and the vector of portfolio weights $\boldsymbol{\omega_K}$ corresponds to the solution for the two optimization problems from which the CTF and CVF are defined. In other words, assuming the vector $\boldsymbol{\omega_J}$ as a portfolio lying on the oval boundary and the vector $\boldsymbol{\omega_{AB}}$ as a portfolio on the linear boundary, $K$ is the only point where the equality $\boldsymbol{\omega_J} = \boldsymbol{\omega_{AB}}$ holds.

From the technical point of view, it is sufficient to apply the following results in order to determine the portfolios lying on the RBF:

1. A portfolio belonging to the CTF (and therefore, to the arc $\widehat{J_0J_1}$) has equation

$$\boldsymbol{\omega_J} = \boldsymbol{\omega_B} - \frac{1}{\lambda_2}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} + \lambda_1\boldsymbol{\iota} + \lambda_3\boldsymbol{\Sigma}\boldsymbol{\omega_B}), \tag{B.1}$$

where

$$\lambda_1 = -\frac{\lambda_3 + b}{a}, \qquad \lambda_2 = -2\sqrt{\frac{d\delta_B}{4T_0\Delta_2 - y^2}}, \quad \text{and} \quad \lambda_3 = -\frac{1}{\Delta_2}\left(\Delta_1 + \frac{y}{2}\lambda_2\right);$$

the parameters $a$, $d$, $\Delta_1$, $\Delta_2$, $\delta_B$ are defined in section 3.1, while $y = \sigma_J^2 - \sigma_B^2 - T_0$ is computed via equation (14). (See Appendix C in Jorion, 2003, for the complete proof)

2. A portfolio that lies on the CVF (the straight line $\mu_P = z_\theta \sigma_P^2 - V_0$ in the $(\sigma_P, \mu_P)$ space) is defined by the linear combination

$$\boldsymbol{\omega_{AB}} = X\,\boldsymbol{\omega_C} + Y\,\boldsymbol{\omega_Q} + Z\,\boldsymbol{\omega_B}, \tag{B.2}$$

where

$$X = \frac{-k_4 + \sqrt{k_4^2 - 4k_3k_5}}{2k_3}, \quad Y = \frac{(k_7 - k_6 X)}{k_6 + ad}, \quad \text{and} \quad Z = 1 - X - Y.$$

The parameters are

$$\begin{aligned}
k_3 &= \frac{k_6^2\sigma_Q^2 + (ad)^2\sigma_B^2 + 2adk_6\mu_B/b}{(k_6+ad)^2} - \sigma_C^2, \\
k_4 &= 2\left[\sigma_C^2 + \frac{-k_6k_7\sigma_Q^2 + adk_8\sigma_B^2 + (k_6k_8 - adk_7)\mu_B/b}{(k_6+ad)^2}\right], \\
k_5 &= \frac{k_7^2\sigma_Q^2 + k_8^2\sigma_B^2 - (2/b)k_7k_8\mu_B}{(k_6+ad)^2} - \left(\frac{V_0 + \mu_{AB}}{z_\theta}\right)^2, \\
k_6 &= b(b - a\mu_B), \\
k_7 &= ab(\mu_{AB} - \mu_B), \\
k_8 &= a(b\mu_{AB} - c),
\end{aligned}$$

where $a$, $b$, $c$ and $d$ are the scalars already defined in section 3.1, $\mu_B$ is the benchmark return, and $\sigma_B^2$, $\sigma_Q^2 = c/b^2$ and $\sigma_C^2 = 1/a$ are the variances of portfolios $B$, $Q$, and $C$. (See Appendix D in Alexander and Baptista, 2008, for the proof).

For any given $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, $\boldsymbol{\omega_B}$ and $T_0$, each point $K \in$ RBF has the property that the vectors in equations (B.1) and (B.2) coincide (the correspondive Value-at-Risk $V_0$ is a result of the optimization described in subsection 3.2).

A special situation arises for portfolio $M$, which lies on the Mean-Variance Frontier. In this case the Mutual Fund Separation Theorem (Merton, 1972) estabilishes that, for any portfolio $P$ lying on the MVF, the portfolio weights can be conveniently obtained via the linear combination

$$\boldsymbol{\omega_P} = K\,\boldsymbol{\omega_Q} + (1 - K)\,\boldsymbol{\omega_C}, \tag{B.3}$$

where $K = \dfrac{\mu_P - \mu_C}{\mu_Q - \mu_C}$ and $\mu_Q = c/b$ and $\mu_C = b/a$ are the returns of portfolios $Q$ and $C$. Equation (B.3) requires $\boldsymbol{\omega_Q}$ and $\boldsymbol{\omega_C}$ already defined before equation (10) and does not depend of any portfolio variance. For its computation in portfolio $M$ it sufficient to set the target return $\mu_P = \mu_M$.

# Appendix C: Stock returns of the S&P 100

Table C.1: Descriptive statistics (in %)

| | | subsample 1: pre COVID-19 | | | | subsample 2: post COVID-19 | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **Stock name** | **mean** | **st.dev.** | **min** | **max** | **mean** | **st.dev.** | **min** | **max** |
| 1. | 3M Company | -0.018 | 1.584 | -13.860 | 4.229 | 0.146 | 1.708 | -7.288 | 7.352 |
| 2. | Abbott Laboratories | 0.092 | 1.213 | -4.834 | 3.172 | 0.166 | 1.730 | -4.869 | 7.557 |
| 3. | AbbVie Inc. | 0.018 | 1.738 | -17.740 | 3.828 | 0.154 | 1.621 | -5.682 | 7.172 |
| 4. | Accenture | 0.161 | 1.047 | -3.942 | 5.065 | 0.207 | 1.890 | -7.297 | 8.742 |
| 5. | Adobe Inc. | 0.148 | 1.478 | -4.652 | 5.082 | 0.154 | 2.245 | -6.251 | 8.705 |
| 6. | Alphabet Inc. (Class A) | 0.092 | 1.469 | -7.798 | 9.185 | 0.220 | 1.948 | -5.666 | 8.518 |
| 7. | Altria Group | 0.030 | 1.540 | -7.164 | 4.034 | 0.138 | 1.664 | -6.241 | 4.567 |
| 8. | Amazon.com | 0.071 | 1.416 | -5.532 | 4.885 | 0.177 | 2.097 | -7.902 | 7.631 |
| 9. | American Express | 0.107 | 1.062 | -3.758 | 4.407 | 0.200 | 2.904 | -9.515 | 19.380 |
| 10. | American International Group | 0.110 | 1.448 | -9.465 | 6.552 | 0.261 | 3.261 | -11.530 | 12.350 |
| 11. | American Tower | 0.155 | 1.130 | -3.763 | 3.269 | 0.043 | 1.911 | -5.353 | 9.012 |
| 12. | Amgen Inc. | 0.100 | 1.293 | -3.847 | 5.779 | 0.089 | 1.711 | -7.067 | 7.849 |
| 13. | Apple Inc. | 0.245 | 1.634 | -10.490 | 6.610 | 0.253 | 2.289 | -8.345 | 9.956 |
| 14. | AT&T Inc. | 0.132 | 1.107 | -4.429 | 4.192 | 0.041 | 1.541 | -6.326 | 6.963 |
| 15. | Bank of America | 0.142 | 1.435 | -4.803 | 6.912 | 0.240 | 2.694 | -10.580 | 13.270 |
| 16. | Berkshire Hathaway | 0.043 | 0.936 | -5.650 | 2.827 | 0.128 | 1.424 | -7.304 | 5.881 |
| 17. | Biogen | -0.010 | 3.021 | -34.570 | 23.200 | -0.047 | 3.724 | -33.080 | 36.450 |
| 18. | BlackRock Inc. | 0.110 | 1.273 | -4.466 | 3.596 | 0.215 | 1.913 | -8.167 | 7.253 |
| 19. | Boeing Co. | 0.011 | 1.804 | -7.032 | 6.063 | 0.205 | 4.083 | -17.940 | 17.790 |
| 20. | Booking Holdings | 0.068 | 1.578 | -11.610 | 6.352 | 0.210 | 2.730 | -8.732 | 17.190 |
| 21. | Bristol-Myers Squibb | 0.091 | 1.625 | -14.230 | 4.898 | 0.063 | 1.424 | -5.906 | 6.618 |
| 22. | Broadcom Inc. | 0.100 | 1.845 | -9.316 | 7.915 | 0.271 | 2.161 | -6.997 | 7.474 |
| 23. | Capital One Financial Corp. | 0.118 | 1.495 | -6.438 | 6.251 | 0.360 | 3.471 | -12.090 | 15.630 |
| 24. | Caterpillar Inc. | 0.071 | 1.678 | -9.569 | 5.320 | 0.276 | 2.250 | -9.112 | 6.372 |
| 25. | Charter Communications | 0.201 | 1.334 | -2.410 | 13.270 | 0.133 | 1.684 | -7.465 | 6.930 |
| 26. | Chevron Corporation | 0.048 | 1.140 | -5.072 | 4.510 | 0.163 | 2.553 | -8.789 | 10.970 |
| 27. | Cisco Systems | 0.053 | 1.529 | -9.009 | 6.443 | 0.118 | 1.812 | -11.860 | 6.822 |
| 28. | Citigroup Inc. | 0.165 | 1.508 | -5.420 | 5.091 | 0.225 | 3.158 | -14.360 | 11.480 |
| 29. | Colgate-Palmolive | 0.068 | 0.997 | -4.097 | 3.729 | 0.075 | 1.135 | -2.722 | 5.078 |
| 30. | Comcast Corp. | 0.110 | 1.182 | -3.484 | 5.345 | 0.180 | 1.791 | -5.870 | 6.500 |
| 31. | ConocoPhillips | 0.018 | 1.729 | -5.795 | 8.665 | 0.224 | 3.339 | -8.616 | 13.410 |
| 32. | Costco Wholesale Corp. | 0.143 | 0.994 | -2.980 | 4.963 | 0.094 | 1.272 | -5.506 | 5.549 |
| 33. | CVS Health | 0.061 | 1.594 | -8.447 | 7.186 | 0.103 | 1.777 | -9.010 | 5.604 |
| 34. | Danaher Corporation | 0.165 | 1.254 | -3.504 | 8.178 | 0.188 | 1.695 | -7.274 | 5.833 |
| 35. | Dow Inc. | 0.068 | 2.059 | -6.306 | 6.027 | 0.321 | 2.829 | -10.440 | 9.053 |
| 36. | Duke Energy | 0.045 | 0.778 | -2.882 | 2.229 | 0.084 | 1.651 | -4.361 | 7.197 |
| 37. | DuPont de Nemours Inc. | -0.066 | 2.096 | -9.681 | 16.390 | 0.321 | 2.446 | -8.275 | 10.480 |
| 38. | Eli Lilly and Company | 0.061 | 1.289 | -4.231 | 3.258 | 0.121 | 2.248 | -9.533 | 14.570 |
| 39. | Emerson Electric Co. | 0.106 | 1.359 | -3.806 | 3.673 | 0.256 | 2.185 | -8.585 | 8.433 |
| 40. | Exelon | 0.025 | 0.964 | -4.676 | 2.489 | 0.081 | 2.083 | -9.482 | 10.350 |
| 41. | Exxon Mobil Corp. | 0.019 | 1.132 | -4.112 | 3.621 | 0.178 | 2.765 | -9.239 | 11.920 |
| 42. | Facebook, Inc. | 0.160 | 1.710 | -7.802 | 10.270 | 0.218 | 2.379 | -8.683 | 7.994 |
| 43. | FedEx | -0.023 | 2.016 | -13.830 | 5.179 | 0.331 | 2.569 | -8.616 | 11.090 |
| 44. | Ford Motor Company | 0.088 | 1.686 | -7.746 | 10.210 | 0.357 | 2.868 | -10.520 | 11.080 |
| 45. | General Dynamics | 0.053 | 1.193 | -3.576 | 3.248 | 0.133 | 1.867 | -6.199 | 6.333 |
| 46. | General Electric | 0.143 | 2.485 | -11.990 | 11.020 | 0.195 | 3.224 | -12.030 | 13.280 |
| 47. | General Motors | 0.048 | 1.517 | -4.778 | 6.817 | 0.390 | 3.079 | -8.149 | 9.300 |
| 48. | Gilead Sciences | 0.024 | 1.318 | -4.464 | 4.495 | -0.040 | 1.828 | -4.988 | 9.288 |
| 49. | Goldman Sachs | 0.120 | 1.451 | -4.279 | 9.115 | 0.295 | 2.310 | -9.520 | 8.585 |
| 50. | Honeywell | 0.122 | 1.076 | -4.015 | 3.718 | 0.194 | 1.964 | -7.182 | 8.136 |

Table C.1 — *continued from previous page*

| | | subsample 1: pre COVID-19 | | | | subsample 2: post COVID-19 | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **Stock name** | **mean** | **st.dev.** | **min** | **max** | **mean** | **st.dev.** | **min** | **max** |
| 51. | Intel Corp. | 0.102 | 1.686 | -9.422 | 7.788 | 0.074 | 2.535 | -17.720 | 7.644 |
| 52. | International Business Machines | 0.077 | 1.272 | -5.682 | 8.124 | 0.091 | 1.953 | -10.430 | 7.673 |
| 53. | Johnson & Johnson | 0.062 | 1.027 | -6.422 | 2.994 | 0.097 | 1.188 | -4.802 | 4.382 |
| 54. | JPMorgan Chase & Co. | 0.143 | 1.157 | -4.241 | 4.581 | 0.215 | 2.412 | -8.713 | 12.700 |
| 55. | Kraft Heinz | -0.095 | 2.648 | -32.100 | 12.610 | 0.202 | 1.754 | -5.244 | 5.561 |
| 56. | Linde plc | 0.124 | 1.197 | -3.836 | 4.643 | 0.191 | 1.829 | -7.270 | 8.163 |
| 57. | Lockheed Martin | 0.159 | 1.019 | -2.544 | 5.504 | 0.043 | 1.647 | -5.052 | 6.481 |
| 58. | Lowe's | 0.108 | 1.632 | -12.610 | 9.849 | 0.310 | 1.985 | -9.368 | 7.689 |
| 59. | Mastercard | 0.177 | 1.329 | -4.871 | 4.627 | 0.151 | 2.269 | -8.462 | 11.510 |
| 60. | McDonald's | 0.054 | 0.935 | -5.173 | 2.305 | 0.126 | 1.460 | -4.440 | 9.915 |
| 61. | Medtronic plc | 0.106 | 1.019 | -6.662 | 3.193 | 0.112 | 1.933 | -7.094 | 8.918 |
| 62. | Merck & Co. | 0.082 | 1.145 | -4.807 | 3.467 | 0.013 | 1.447 | -5.618 | 5.188 |
| 63. | MetLife Inc. | 0.092 | 1.238 | -4.489 | 3.651 | 0.282 | 2.699 | -9.473 | 10.450 |
| 64. | Microsoft | 0.177 | 1.231 | -3.748 | 4.546 | 0.158 | 1.862 | -6.395 | 7.174 |
| 65. | Mondelēz International | 0.132 | 0.953 | -3.881 | 5.441 | 0.068 | 1.250 | -4.980 | 4.676 |
| 66. | Morgan Stanley | 0.102 | 1.447 | -4.773 | 4.207 | 0.327 | 2.427 | -8.841 | 8.925 |
| 67. | Netflix | 0.073 | 2.152 | -10.840 | 9.279 | 0.126 | 2.661 | -8.985 | 15.580 |
| 68. | NextEra Energy | 0.146 | 0.882 | -3.431 | 3.058 | 0.095 | 1.902 | -9.767 | 5.644 |
| 69. | Nike, Inc. | 0.125 | 1.290 | -6.842 | 4.588 | 0.185 | 1.886 | -7.930 | 8.399 |
| 70. | Nvidia Corporation | 0.213 | 2.532 | -14.880 | 7.002 | 0.271 | 2.878 | -9.736 | 9.568 |
| 71. | Oracle Corporation | 0.068 | 1.226 | -4.357 | 7.864 | 0.149 | 1.545 | -6.754 | 6.434 |
| 72. | PayPal | 0.090 | 1.574 | -5.229 | 8.211 | 0.357 | 2.788 | -9.303 | 13.110 |
| 73. | PepsiCo | 0.098 | 0.860 | -2.834 | 3.689 | 0.074 | 1.270 | -4.803 | 5.579 |
| 74. | Pfizer Inc. | -0.024 | 1.186 | -6.632 | 3.089 | 0.076 | 1.669 | -7.574 | 7.411 |
| 75. | Philip Morris International | 0.113 | 1.506 | -8.075 | 7.897 | 0.098 | 1.589 | -6.144 | 4.595 |
| 76. | Procter & Gamble | 0.132 | 1.022 | -3.952 | 4.750 | 0.089 | 1.158 | -4.372 | 4.534 |
| 77. | Qualcomm | 0.180 | 2.391 | -11.490 | 20.870 | 0.266 | 2.650 | -9.244 | 14.170 |
| 78. | Raytheon Technologies | 0.134 | 1.291 | -4.553 | 5.244 | 0.112 | 2.784 | -8.404 | 14.220 |
| 79. | Salesforce | 0.070 | 1.594 | -5.402 | 5.636 | 0.148 | 2.708 | -8.908 | 23.150 |
| 80. | Simon Property Group | -0.020 | 1.116 | -4.105 | 4.499 | 0.300 | 4.438 | -15.380 | 24.580 |
| 81. | Southern Company | 0.163 | 0.868 | -2.554 | 2.807 | 0.070 | 1.769 | -7.675 | 9.717 |
| 82. | Starbucks Corp. | 0.128 | 1.189 | -4.435 | 8.559 | 0.202 | 1.953 | -8.502 | 7.249 |
| 83. | T-Mobile US | 0.071 | 1.268 | -4.372 | 5.289 | 0.154 | 1.853 | -6.395 | 9.574 |
| 84. | Target Corporation | 0.266 | 1.971 | -5.818 | 18.590 | 0.297 | 1.783 | -7.011 | 11.910 |
| 85. | Tesla, Inc. | 0.116 | 3.042 | -14.630 | 16.270 | 0.710 | 4.739 | -23.650 | 17.930 |
| 86. | Texas Instruments | 0.129 | 1.663 | -7.776 | 7.174 | 0.255 | 2.021 | -5.305 | 8.601 |
| 87. | The Bank of New York Mellon | 0.032 | 1.412 | -10.010 | 5.538 | 0.143 | 2.251 | -7.545 | 11.030 |
| 88. | The Coca-Cola Company | 0.076 | 1.062 | -8.813 | 5.895 | 0.080 | 1.590 | -6.544 | 6.278 |
| 89. | The Home Depot | 0.102 | 1.131 | -5.592 | 4.301 | 0.197 | 1.587 | -6.068 | 6.829 |
| 90. | The Walt Disney Company | 0.114 | 1.378 | -5.067 | 10.930 | 0.248 | 2.323 | -8.130 | 12.750 |
| 91. | Thermo Fisher Scientific | 0.153 | 1.397 | -6.144 | 5.518 | 0.183 | 1.888 | -8.432 | 7.438 |
| 92. | U.S. Bancorp | 0.107 | 1.010 | -4.389 | 2.876 | 0.197 | 2.877 | -8.433 | 11.990 |
| 93. | Union Pacific Corporation | 0.113 | 1.518 | -6.243 | 8.367 | 0.179 | 1.810 | -7.171 | 6.482 |
| 94. | United Parcel Service | 0.085 | 1.504 | -8.476 | 8.308 | 0.241 | 2.054 | -9.228 | 13.430 |
| 95. | UnitedHealth Group | 0.079 | 1.559 | -5.321 | 7.844 | 0.159 | 1.853 | -7.475 | 9.835 |
| 96. | Verizon Communications | 0.051 | 0.986 | -4.462 | 3.667 | 0.047 | 1.103 | -3.218 | 5.112 |
| 97. | Visa Inc. | 0.136 | 1.121 | -4.945 | 4.218 | 0.107 | 1.894 | -6.439 | 10.960 |
| 98. | Walgreens Boots Alliance | -0.043 | 1.718 | -13.700 | 5.605 | 0.087 | 2.468 | -10.130 | 7.213 |
| 99. | Walmart | 0.101 | 0.889 | -3.327 | 5.932 | 0.075 | 1.400 | -6.700 | 6.559 |
| 100. | Wells Fargo | 0.068 | 1.227 | -4.426 | 3.696 | 0.130 | 3.104 | -10.350 | 10.020 |
| *B* | S&P 100 | 0.098 | 0.797 | -3.106 | 3.487 | 0.160 | 1.310 | -5.853 | 6.438 |

# Appendix D: the Constrained Mean-TEV Frontier and the Fixed VaR-TEV Frontier

This Appendix contains a brief geometrical illustration of the scenarios for the Constrained Mean-TEV Frontier (CMTF) by Alexander and Baptista (2008) and the FVTF by Palomba and Riccetti (2012) arising from different VaR restrictions. The Constrained Mean-TEV Frontier (CMTF) is a portfolio boundary satisfying the VaR constraint, while the TEV is kept at its minimum value (Roll, 1992). The FVTF considers the CTF as well.

In general, based on different VaR bounds, the CMTF could be

(a) an empty set if the CVF lies to the left of the MVF, *i.e.* in the set of inadmissible portfolios;

(b) a single admissible portfolio when the CVF is tangent to the MVF;

(c) a segment, if the CVF crosses the MVF but not the MTF (but may be tangent to the MTF);

(d) an arc, consecutive to two segments, if the CVF crosses both the MVF and the MTF.

Assigning a VaR bound to each scenario, the FVTF could be

**An empty set** in three cases:

- in case (a) above,
- in case (b) above, when the tangency between the CVF and the MVF does not satisfy the TEV restriction,
- in case (c) above, when the CVF crosses the MVF only.

**A single portfolio** when the CVF is tangent to the CTF (portfolio $K$ where TEV $= T_0$ and VaR $= V_0$)

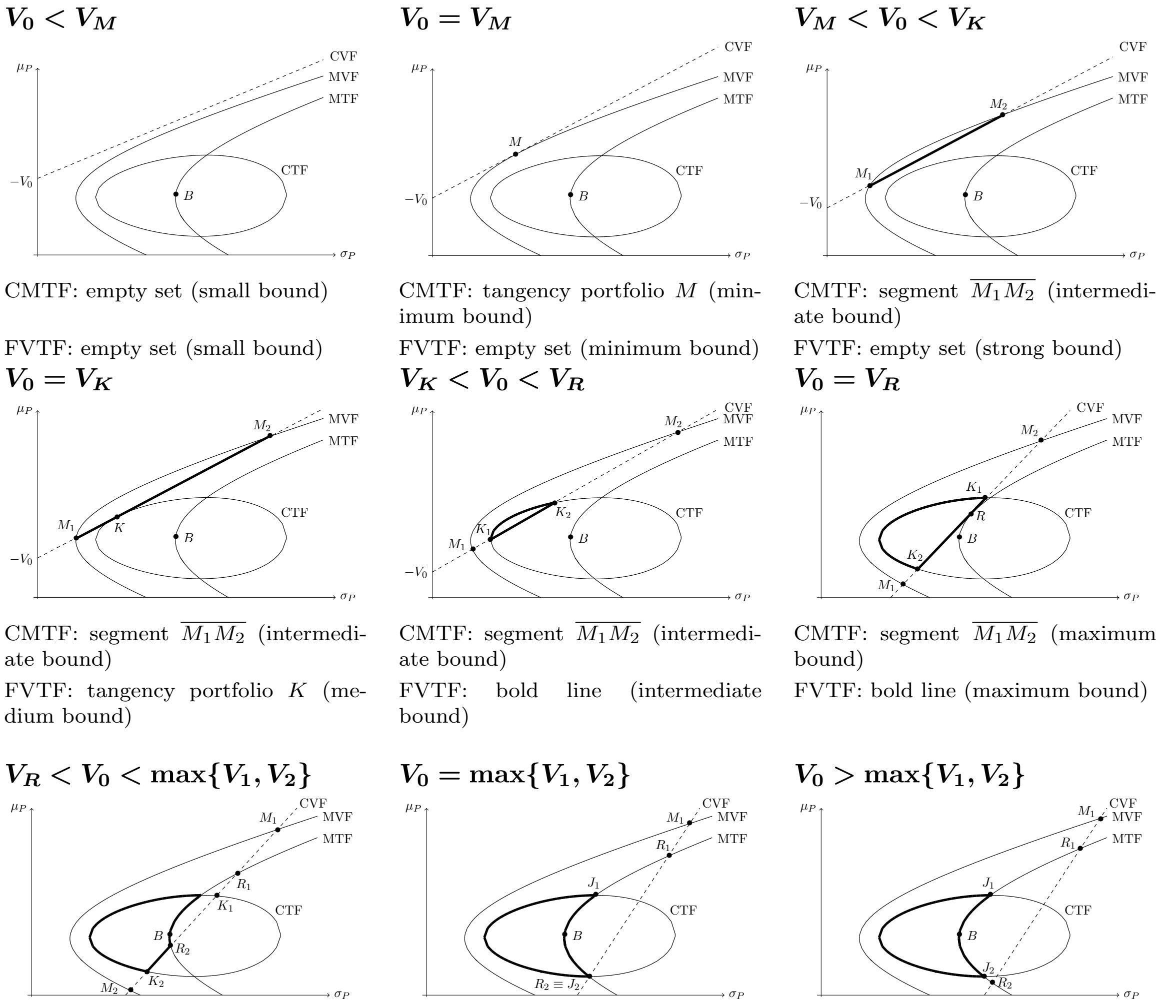
**A closed and bounded set** in several cases:

- when the CVF crosses the CTF given by the segment and the left arc identified by the two intersections;

- when the CVF crosses all the other frontiers. This boundary have a horseshoe-like shape: the linear frontier crosses both the CTF and the MTF thus defining two arcs on the left whose endpoints are joined by two segments;
- same as above, but the straight line passes through (at least) one intersection between the oval and the hyperbolic boundaries;
- where the set is given by the two left arcs defined by the two intersections between the CTF and the MTF. In this case the VaR limit is not binding.

Clearly, an empty FVTF indicates that no admissible portfolio can satisfy the joint limits on risk indicators, so these limits are incompatible. Otherwise, inside this frontier, the asset manager has to face a trade-off: she can try to reduce the VaR via a riskier active strategy that enlarges the TEV or try to reduce the relative risk, but this implies a higher VaR near to that of the selected benchmark. Figure D.1 shows the various scenarios in the $(\sigma_P, \mu_P)$ space.

Figure D.1: CMTF and FVTF for different VaR bounds




**$V_0 < V_M$**

CMTF: empty set (small bound)

FVTF: empty set (small bound)

**$V_0 = V_M$**

CMTF: tangency portfolio $M$ (minimum bound)

FVTF: empty set (minimum bound)

**$V_M < V_0 < V_K$**

CMTF: segment $\overline{M_1M_2}$ (intermediate bound)

FVTF: empty set (strong bound)

**$V_0 = V_K$**

CMTF: segment $\overline{M_1M_2}$ (intermediate bound)

FVTF: tangency portfolio $K$ (medium bound)

**$V_K < V_0 < V_R$**

CMTF: segment $\overline{M_1M_2}$ (intermediate bound)

FVTF: bold line (intermediate bound)

**$V_0 = V_R$**

CMTF: segment $\overline{M_1M_2}$ (maximum bound)

FVTF: bold line (maximum bound)

**$V_R < V_0 < \max\{V_1, V_2\}$**

CMTF: segment $\overline{M_1R_1}$, arc $\widehat{R_1R_2}$, segment $\overline{R_2M_2}$ (large bound)

FVTF: bold line (large bound)

**$V_0 = \max\{V_1, V_2\}$**

CMTF: segment $\overline{M_1R_1}$, arc $\widehat{R_1R_2}$, segment $\overline{R_2M_2}$ (large bound)

FVTF: bold line (larger bound)

**$V_0 > \max\{V_1, V_2\}$**

CMTF: segment $\overline{M_1R_1}$, arc $\widehat{R_1R_2}$, segment $\overline{R_2M_2}$ (large bound)

FVTF: bold line (no bound)

# Appendix E: Supplementary material

All the data and the routines for the analysis carried out in this article can be found at

`http://utenti.dea.univpm.it/palomba/TEVaR/`