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A STOCHASTIC MODEL OF WEALTH ACCUMULATION WITH CLASS DIVISION

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Abstract In this paper we propose a stochastic model in which wealth accumulation depends on the role that agents play in the society: capitalists or workers. A random mechanism of class selection shapes the social structure of the economy based on wealth distribution dynamics. As a result, the society may evolve towards an unequal outcome with few rich and many poor individuals, even starting from perfect equality. We study the dynamic properties of the model by means of computer simulations. A maximum likelihood estimation procedure is applied to analyse the Pareto or power law tail of wealth distribution. We also provide a scenario analysis to explore the system's behaviour under alternative parameter settings.

Keywords: wealth distribution; power law; social classes.

JEL Classification Numbers: P10, D31, C63.

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1 Introduction

Are richer people smarter? According to Levy (2003) the answer to this question seems to be negative, at least when the analysis is focussed on the process of wealth accumulation due to capital investments. What he found is that, above a certain threshold, wealth accumulation is the result of a multiplicative stochastic process leading to a power law (or Pareto) distribution of the "fat" right tail. Levy (2003) maintained that this result depends on the assumption that agents are *homogeneous* with respect to their investment talent. When *heterogeneous* abilities are considered, results which are inconsistent with the empirically observed Pareto distribution emerge. Then, the author concludes that wealth inequality *for the high-wealth range* seems to be the result of *chance* rather than *differential abilities* so contrasting the original interpretation of the factors underlying the Pareto distribution, given by Pareto himself, that wealth inequality is related to a skew distribution of human abilities.

According to our vision, there is a relevant factor which contributes to the explanation of wealth inequality – for the entire range of the distribution – and of its interplay with growth: the working of an economic system is characterised by *class division*, as noted by Classical economists and particularly by Karl Marx. Then, the wealth accumulation process proceeds at a different pace for capitalists and workers and, under certain conditions, this may lead to a highly unequal distribution of wealth and a persistent division in social classes according to which agents which spend a long time as a capitalist are more likely to become richer and richer, while a large majority of the population remains in the working class, with a small probability to access the capitalist role.

Our aim is then contributing to the literature on income/wealth inequality by stressing the role of class division in shaping the wealth accumulation process. In this paper we propose a stochastic model in which wealth accumulation depends on the role that agents play in the society: capitalists or workers. A stochastic process of class selection shapes the social structure of the economy as a consequence of wealth distribution dynamics. As a result, the society may evolve towards an unequal outcome with few rich and many poor individuals, even starting from perfect equality.

In our artificial economy, capitalists earn a profit generated by a stochastic multiplicative process; instead, workers earn a wage generated by an additive stochastic process. This is similar to the assumption on the income generation processes made by Levy (2003) – who investigated the properties of a wealth accumulation process supported by financial investments and the conditions under which a power law distribution emerges – and Nirei and Souma (2007) – which, starting from Levy (2003) and introducing a labour income process, proposed a framework to fit the empirical distribution of income in a "two factors model". In this setting, we introduce a random process of class selection according to which richer agents are more likely to become capitalists, and viceversa.

In this perspective, it is *power* rather than *chance* to play a central role in shaping economic and social dynamics. In our simplified stochastic framework, we refer to power as *implicitly* resulting from owning a "large" wealth that, in turn, increases the probability of becoming (or remaining) a capitalist and accumulating more and more wealth. We model such an *amplification mechanism* by introducing a stochastic rule of class selection related to wealth accumulation dynamics: a stable social configuration with few capitalists and many workers may emerge, possibly leading to a persistent class division with low mobility between social classes. In our framework, then, we could say that "richer people are more powerful" in managing and remaining rich, due to the characteristics of the wealth accumulation process itself.[1]

The remainder of the paper is organised as follows. In Section 2, we discuss the role of a stochastic multiplicative process, and its interplay with a reflective lower bound, in explaining the emergence of an unequal wealth distribution, as in the case of a power law tail for the high-wealth range. This is the framework from which we develop our analysis incorporating a stochastic mechanism of class division. In Section 3, we present our baseline model, from its setup to agents' class behaviour, from the stochastic processes of income generation to the agents' rules for consumption, saving and wealth accumulation. In Section 4, we discuss the results coming from the simulation of the baseline model for which initial conditions and the parameter setting are provided. After that, we explore the effects of changing the parameter setting and discuss the implications of the model findings. Finally, in Section 5, we provide some concluding remarks suggesting possible directions for future research.

2 Related literature

In this section we provide a discussion on the previous literature related to our contribution. An interesting research area is aimed at investigating the interplay between income inequality and wealth accumulation and, in particular, the emergence of a power law tail in income/wealth distributions. There are many contributions in this research field, at least starting from Pareto (1897), of both theoretical and empirical nature. In what follows, we will focus on Levy (2003) and Nirei and Souma (2007) which provide a stochastic framework that we will further extend by incorporating a mechanism of class division.

[1] Some words of caution are in order about the meaning of *power* in our context. For sure, we refer to the concept of power only in a very stylised and abstract way, given that we do not explicitly model power relations between capitalists and workers, nor give details of the social and economic networks underlying them. As it should be clear in the following, we just focus on the self-reinforcing process of wealth accumulation when class division characterises the economic system. In a sense, in our simple stochastic model a large wealth gives an individual, which plays the role of capitalist for enough time, the *power* to accumulate an even greater wealth.

As said above, we assume that capitalists' profits are generated by a stochastic multiplicative process, while workers' wages by an additive one. This is similar to what is assumed by Nirei and Souma (2007) in order to fit the empirical distribution of income for US and Japan in the period 1960-1999. Let's then describe the main aspects of such a framework.

They start from the observation that "[s]ince Pareto, it has long been known that the tail of distribution of income or wealth w universally obeys a power-law distribution $w^{-\alpha}$ for a constant α around $1.5 - 2.5$. A multiplicative process of wealth accumulation has been a standard explanation for the heavy tail". (Nirei and Souma, 2007, p. 440).[2] A first relevant contribution in this direction was the Gibrat's "law of proportionate effect", according to which the multiplicative process leads to a lognormal distribution if the rates of income growth are stochastic and independent on the initial size (Sutton, 1997). Successively, many authors noted that a slight modification of the process leading to a lognormal distribution may generate a power law (Champernowne, 1953; Simon, 1955; Rutherford (1955); Reed, 2001). Recent contributions in this direction have stressed the role of a "reflective multiplicative process" or of a similar mechanism based on a "Kesten process" in explaining the evolution of the distribution tail (Levy and Solomon, 1997; Sornette and Cont, 1997; Takayasu et al., 1997).

The role of a reflective lower bound and its interplay with a multiplicative process have been investigated by Levy (2003). He proposed a stochastic model of wealth accumulation by financial investments and analised the conditions under which the system converges to the empirically observed Pareto wealth distribution (Pareto, 1897). The Pareto distribution is given by the following probability density function:

$$P(\omega) = \xi \omega^{-(1+\alpha)} \quad for \;\; \omega \geq \omega_0$$

where ω stands for wealth, ω_0 is the lower end of the high-wealth range (that is the threshold above which an agent obtains a stochastic return from a financial investment), ξ and α are positive parameters. The stochastic multiplicative process for wealth accumulation is given by:

$$\omega_i(t+1) = \omega_i(t)\tilde{\lambda}$$

where $\omega_i(t)$ is the wealth of investor i at time t, and $\tilde{\lambda}$ is the total return, which is a random variable drawn from some distribution $f_i(\lambda)$. The sub-index i in $f_i(\lambda)$ means that each investor may have a different distribution of returns on his investment. Levy (2003) proposes two cases: (i) *homogeneous investment talent*, according to which $f_i(\lambda) = f(\lambda)$ for all investors; (ii) *heterogeneous investment talent*, according to which investors with

[2] Moreover, "[t]his explanation makes good economic sense because the rate of return for asset is a stationary process" (Nirei and Souma, 2007).

diverse talents face different return distributions. It is worth to note that only in the case (i) a power law for the high-wealth range emerges, while the resulting distribution of wealth in the case (ii) is inconsistent with the empirical evidence.[3]

The model under scrutiny is aimed at investigating only top income dynamics for which the power law characterises the "fat" right tail of the distribution. In other words, the author confined the analysis to wealth levels larger than ω_0, maintaining that the wealth of individuals in the high-wealth range typically changes mainly due to capital investments, while the main factors influencing the wealth at the lower range are usually labour income and consumption (which are basically additive rather than multiplicative). According to Levy (2003), there are many ways to model the boundary between the high-wealth range and the lower one. The author assumed that only those people with wealth exceeding ω_0 participate in the stochastic multiplicative investment process. Then, for all individuals i and for all times t, the following condition holds: $\omega_i(t) \geq \omega_0$.[4]

Levy (2003) introduced his analysis maintaining that "[i]n ancient Greece only the wealthy land owners had voting rights. The logic behind this law was that voting privileges should be given only to the wise citizens, and wealthy people have proven their wisdom by becoming (and managing to remain) rich. The idea that the rich are rich because they are smarter has been with us ever since, and has become a central part of modern western ideology and culture" (p. 43). On the basis of model findings, he concluded "that the wealth inequality *at the high-wealth range* is not likely to induce efficient capital investment by investors in this range" (p. 59). Accordingly, his main result is that the model is able to replicate the empirically observed Pareto distribution for the high-wealth range under the condition that investors' abilities are *homogeneous*. It is worth to note that, when abilities are *heterogeneous*, the model no longer converges to a power law tail, suggesting that in this framework the high inequality of wealth distribution is mainly due to *chance* rather than *differential abilities*.[5] Interestingly enough, this contrasts the original interpretation of the factors underlying the Pareto distribution provided by Pareto himself that the distribution of wealth correspondes to a distribution

[3]According to Levy (2003), which simulated his model for different forms of the return distribution, different distributions of talent, with investors having finite lifespans and inherited wealth, etc., "it seems that any stochastic multiplicative wealth accumulation model which assumes even a mild degree of differential investment talent leads to a distribution of wealth which is inconsistent with the empirical Pareto distribution" (p. 56).

[4]When individuals' wealth changes, they may cross the boundary between the upper and the lower wealth region. Given that the author does not model wealth dynamics in the lower range, he simply assumes that the flow across the boundary is equal in both directions, that is the number of people involved in the stochastic multiplicative process is constant. See Levy (2003), pp. 48-49, for further details on the definition of the lower bound.

[5]"The result of this paper does not mean that *only* luck matters, and that any investment strategy is as good as any other. On the contrary, it means that one must apply his investment skills just in order to have a fair chance in the competition with other investors. Our findings suggest that because investors in the high-wealth range seem to have similar investment talents, at the margin it is only luck that differentiates between them" (Levy, 2003, p. 58).

of human abilities (even though he did not provide a model to explain such a complex nexus).

Nirei and Souma (2007) extended the Levy's framework by incorporating labour income dynamics in the wealth accumulation stochastic process. In other words, they extend the analysis to the low-wealth range that Levy (2003) did not explicitly model. In the Nirei-Souma model, agents benefit from asset accumulation as well as from labour income in the following way:

$$a(t+1) = \lambda(t)a(t) + w(t) - c(t)$$

where λ is a stochastic variable (independently and identically distributed across individuals and time), a is the asset, w is the wage, c is consumption, and t is time. Accordingly, $w(t) - c(t)$ are savings from labour income, that have the role of the reflective barrier for asset accumulation in the model. Consumption $c(t)$ is assumed to be a linear function in asset and disposable income: $c(t) = \bar{w}(t) + b(a(t) + w(t) - \bar{w}(t))$, where b is the marginal propensity to consume, and $\bar{w}(t)$ is the subsistence income, that is the minimum level of consumption.[6]

The authors assumed that the log return – $log\lambda(t)$ – follows a normal distribution, that is that the asset return is stationary in continuous time. Instead, labour income evolves as an additive process with a "reflective lower bound" given by a minimum wage.[7] As a result, the multiplicative process of asset accumulation generates a power-law distribution, while the additive process of wage dynamics generates an exponential decay. In this way, Nirei and Souma (2007) extend the result reached by Levy (2003), adding to the power law tail for the high-wealth range, the exponential decay for the low-wealth range. All in all, this two factors model leads to an interpretation of the whole distribution as explained by two regimes, with a discontinuity at the boundary: following Draculescu and Yakovenko (2001), the whole distribution is described by an exponential distribution in the range from low to middle income and a Pareto distribution for the top portion.[8]

Then, in the Nirei-Souma model, agents accumulate wealth due to both labour and asset incomes (where savings from the labour income represent the lower bound for the multiplicative process of asset accumulation). According to our interpretation, this is

[6]As maintained by the authors, the linearity in the consumption function is a crude assumption. They also maintained that their simulation results are robust to introducing concavity for lower-middle incomes, while this affects the results for top incomes.

[7]See Nirei and Souma (2007), pp. 449-450, for further details on the definition of the lower bound as well as on the charactristics of the stationary process of asset returns.

[8]Instead of employing two different distributions for low-middle (exponential) and high (power-law) incomes, Clementi et al (2010) proposed a κ-generalised distribution as a descriptive model for the distribution and the dispersion of income within a population based on deformed exponential and logarithm functions. Using the Italian personal income data from 1986 to 2006 the authors show that this three-parameters distribution is able to describe the entire income range, including the Pareto upper tail. The same authors successfully applied the κ-generalised distribution to fit personal income data for Great Britain, Germany and USA (Clementi et al, 2011).

somewhat reminiscent of the Marginalist tradition and the Walrasian general economic equilibrium approach for which the capital employed in firms to produce goods and services, in combination with labour, belong to all individuals (although in different parts). In this setting the effort of agents as an enterpreneur or a worker is remunerated according to their marginal contribution to production (a condition which holds for all individuals in the society). This should lead to a *harmonious* society in which agents' claims on "how to divide the cake" are compatible (and economic growth is the result of decentralised decisions taken by atomistic operators in free markets which should be efficient both statically and dynamically).

Contrary to this, and according to a *classical political economy* vision, growth and distribution are related to social structure and it is not equivalent in order to accumulate wealth to be an enterpreneur or a worker. This leads to a *conflictual* society in which a class represents the "capital" and the other class represents "labour", agents' claims on income/wealth distribution being often not compatible. According to this perspective, in developing our own framework we will follow the Nirei-Souma model in assuming that capital remuneration is given by a multiplicative stochastic process, while labour remuneration by an additive one, but we further assume that agents no longer obtain both capital returns and labour income, given that they split up in two classes, with capitalists gaining profits and workers gaining wage, so having different opportunities in accumulating wealth.

As said above, in the Levy model only agents with a wealth larger than a minimum level obtain stochastic returns from capital investment. In a sense, then, and as we will further explain, this is somewhat closer to our assumption that wealth accumulation depends on class division. Anyway, Levy (2003) did not provide an explicit analysis of the transition of individuals from the low-middle to the high range – assuming that the inflow equals the outflow so that the number of individuals in the high range is stationary. This leads to a context in which we observe a power law distribution for the high-wealth range without following the individual stories leading to richness or poorness. As a consequence, the model does not analyse the mechanisms underlying the possibility that a "poor" individual becomes "rich", and viceversa, or the conditions under which the poor remain poor and the rich become even richer.

By modelling both the high-wealth range and the middle-low one, as in the Nirei-Souma model but also introducing class division, we will investigate these transitional dynamics – that is the continuous inflows and outflows from and to different classes – and the conditions under which a small class of capitalists persistently gains profits from capital investments, so boosting wealth accumulation, while the accumulation of wealth proceeds at a lower pace for a large working class which gains wages as labour income. So we will analyse the dynamic properties of an artificial society in which richer individuals are more "powerful" than poorer ones – even having homogeneous abilities and an equal initial endowment – in reaching the goal of accumulating wealth, because of the different time spent as capitalists rather than as workers. We will also investigate the relationship between class division and inequality by applying a maximum likelihood estimation procedure to the analysis of the Pareto or power law tail of wealth distribution.

3 The model

3.1 Setup

Consider an artificial society composed of a multitude of agents, indexed by $i = 1, 2..., N$, each of which has an initial endowement of wealth equal to ω_0. Agents have the goal of accumulating wealth whose evolution depends on a stochastic process we are going to describe. The agents are *homogeneous* in their abilities (or talent). Instead, they may become (highly) heterogeneous with respect to the owned wealth.

In each period of time, $t = 1, 2, ...T$, the generic agent i can belong to one of the two *social classes*: capitalists and workers. We employ the index k to refer to an agent as a capitalist, while w for a worker. N_t^k and N_t^w indicate the number of capitalists and workers at time t, respectively. Accordingly, capitalists are indexed by $k = 1, 2, ..., N_t^k$, and workers by $w = 1, 2, ..., N_t^w$.

Each agent i, at each time t, is characterised by the following triple: $\{\omega_t^i, \epsilon_t^i, s_t^i\}$, where ω_t^i is individual wealth, ϵ_t^i is a random shock – independently and identically distributed across individuals and time –, and s_t^i is the status or the role played by the agent i at time t in the society – then, s is equal to k for a capitalist and w for a worker.

In a nutshell, period after period agents split up in two social classes according to a class division rule (which affects the variable s); agents who play the role of capitalists invest a fraction of their wealth in a risky process; other agents are workers; both profits and wages depend on stochastic processes (and, then, on the realisation of the random shock ϵ); the accumulation of wealth (that is ω) proceeds at a rate and with a degree of inequality that depend on model assumptions and parameters. In what follows we will describe the details of this stochastic process of wealth accumulation with class division.

3.2 Class division rule

In this economy, whether an agent will be a capitalist or a worker in period t depends on a random process related to the distribution of wealth in the society. We build the following individual variable:

$$p_t^i = \omega_{t-1}^i / \omega_{t-1}^{max}$$

where ω_t^{max} is the maximun value of agents' wealth in period $t-1$.[9] For each agent i, the value of another variable, $\tilde{p}_t^i$, is picked at random from a uniform distribution in the interval (0,1). Then, the *role* played by the agent i at time t depends on the following stochastic process of class division:

$$s_t^i = \begin{cases} k & \text{if } p_t^i > \tilde{p}_t^i \\ w & \text{otherwise.} \end{cases}$$

Accordingly, in this artificial society, it is more likely that a richer agent will be a capitalist (indexed by k), while a poorer agent a worker (indexed by w).

3.3 Agents and incomes

Capitalists start the *risky* economic activity with the aim of producing a profit and accumulate wealth. Let's assume that a capitalist – say the agent k – invests a fraction γ of its accumulated wealth:

$$\bar{\omega}_t^k = \gamma \omega_{t-1}^k \tag{1}$$

where $0 < \gamma < 1$.[10] Then, total wealth invested in period t is

$$\bar{\Omega}_t = \sum_{k=1}^{N_t^k} \bar{\omega}_t^k. \tag{2}$$

According to what said above, capitalist k's profit at time t is the result of a *multiplicative stochastic process*:

$$\pi_t^k = \phi \bar{\omega}_t^k \tag{3}$$

where ϕ is a normally distributed random shock with positive mean μ_ϕ and finite standard deviation σ_ϕ, identically and independently distributed across capitalists and time.[11] Aggregate profits are:

[9]Accordingly, the individual variable p_t^i has a zero lower bound – in the limit case of an agent with no wealth at all – and a unitary upper bound – for the richest agent(s).

[10]Then, we made the strong assumption that a capital market – allowing capitalists to invest more than their own wealth – is not considered. The same holds for previous contributions like the ones by Levy (2003) and Nirei and Souma (2007). However, considering that ω represents the total wealth of an individual, it is quite reasonable that one invests only a fraction of it in a risky process. In a sense, the fraction of non-invested wealth can be implicitly considered as non-liquid assets. See below for further comments on the topic. Anyway, this is one of the hypothesis we will remove in the next future to improve the modelling framework.

[11]As in the literature cited in Section 2, we assume that returns are independent of asset size. In our framework this means that the individual profit rate is independent of the capital invested in the risky process. Obviously, the absolute level of the profit that makes individual wealth growing through accumulation depends on the investment. Moreover, in this version of the model we do not distinguish between *real* and *financial investments*; accordingly, profits can derive from both real and financial

$$\Pi_t = \sum_{k=1}^{N_t^k} \pi_t^k. \tag{4}$$

Then, we define the profit rate as:

$$\pi_t^r = \Pi_t / \bar{\Omega}_t. \tag{5}$$

For the reasons explained in Section 2, we assume that the wage paids to the j^{th} worker at time t is the result of an *additive stochastic process*:

$$w_t^j = w_{t-1}^j + \epsilon \tag{6}$$

where ϵ is a normally distributed random shock with a positive mean μ_ϵ and a finite standard deviation σ_ϵ, identically and independently distributed across workers and time.[12]

So, the wage bill for the whole economy is:

$$W_t = \sum_{j=1}^{N_t^j} w_t^j. \tag{7}$$

3.4 Consumption, saving and wealth accumulation

We assume that the agent i's consumption function is given by:

$$c_t^i = \begin{cases} \bar{c} + c'(w_t^i - \bar{c}) + c''' \sqrt{\omega_{t-1}^i} & \text{if } s_t^i = w \\ \bar{c} + c''(\pi_t^i - \bar{c}) + c''' \sqrt{\omega_{t-1}^i} & \text{otherwise.} \end{cases} \tag{8}$$

where $\bar{c}$ is the *minimum consumption* level,[13] $c' > 0$ is the propensity to consume the labour income, $c'' > 0$ is the propensity to consume the profit income,[14] and c''' represents the propensity to consume a fraction of wealth.[15] Then, aggregate consumption is:

activities we do not model explicitly. Furthermore, the hypothesis that the random profit is normally distributed is made for the sake of simplicity. More specific assumptions on the stochastic process can be made in order to improve the modelling framework and have a more realistic model to replicate the empirical evidence.

[12]In each period, we compute the wage for all agents, then also for non-worker agents (which is just a potential labour income), so that if the worker j was a capitalist in period $t-1$, then w_{t-1}^j represents the wage the agent would have earned as a worker; this is used as the base to compute the current wage. Moreover, a zero lower bound holds for wages, given that these cannot be negative. As in the case of profits, the hypothesis that the random wage is normally distributed is made to simplify the modelling framework and more realistic assumptions on the stochastic process can be introduced to improve it.

[13]In the next Section we will also consider a time-varying minimum amount of consumption as a fraction of average consumption.

[14]If the profit realised by the capitalist k is negative in period t, then its consumption due to the profit income is set equal to zero.

[15]It is worth to note that we introduce concavity in the term representing wealth consumption. Accordingly, the rich spend more than the poor in absolute terms but less in proportion to their wealth.

$$C_t = \sum_{i=1}^{N} c_t^i. \tag{9}$$

As time elapses, agents' wealth evolves as the result of the accumulation of non-consumed income, that is savings. The i^{th} agent's wealth in period t depends on the wage or profit income net of actual consumption:

$$\omega_t^i = \begin{cases} \omega_{t-1}^i + w_t^i - c_t^i & \text{if } s_t^i = w \\ \omega_{t-1}^i + \pi_t^i - c_t^i & \text{otherwise.} \end{cases} \tag{10}$$

Finally, by summing up individual wealth we obtain the society's aggregate wealth:

$$\Omega_t = \sum_{i=1}^{N} \omega_t^i. \tag{11}$$

4 Simulations

4.1 Initial conditions and parameter setting

We study the dynamic properties of this artificial society by means of computer simulation. Let's set the initial endowement $\omega_0 = 100$ and the initial wage level $w_0 = 1$, the same for all agents. We investigate the properties of a society composed of $N = 1000$ agents for a time span of $T = 1000$ periods.[16] The parameters are set as follows: $\gamma = 1/2$, $\bar{c} = 1$, $c' = 0.8$, $c'' = 0.2$ (hence, following Kaldor, 1955: $c' > c''$); $c''' = 0.005$.[17] The mean of the additive process for wages as well as of the multiplicative process for profits is $\mu_\epsilon = \mu_\phi = 0.05$. The standard deviation of the two stochastic processes is $\sigma_\epsilon = \sigma_\phi = 1/2$.

In what follows, we firstly analyse the behaviour of the *baseline* model, then we investigate the effects of changing the parameter setting. In particular, we will analyse the role of parameters regarding consumption/saving behaviour, the fraction of capital invested by capitalist in the risky process, and the income generation processes. We also propose some additional experiments by introducing a positive return on money not invested in the stochastic process, to consider the role of social conflict in shaping macroeconomic dynamics, and to test the dependence of model findings on initial conditions.

According to Stiglitz (1969), capital accumulation is independent of wealth distribution only if we assume linearity in the consumption function. By contrast, in our model the non-linearity in the consumption function leads to a dependence of accumulation on how wealth is distributed among individuals.

[16] The statistical and graphical results we will present in the following refer to simulation results for $t > 500$. This choice was done in order to get rid of transitional dynamics due to the effect of initial conditions.

[17] As said above, wealth accumulation depends on wealth distribution because of non-linearity in the consumption function of agents. See Stiglitz (1969) for a parallel with an economic growth setting, according to which class savings behaviour is a "force for inequality" in the distribution of wealth.

4.2 The dynamics of the baseline model

Let's now investigate the system's behaviour when the model is simulated for the parameter setting and the initial conditions set above. Starting from perfect equality, then with a zero Gini index at time $t = 0$, its average value over the time span for which statistics are gathered ($500 < t \leq 1000$) is almost 0.37. Figure 1, upper-left panel, shows that wealth distribution evolves towards an unequal outcome with few rich and many poor individuals, that is a right-skew heavy-tail distribution. Then, wealth distribution seems to evolve towards a power-law or Pareto distribution for the high-wealth range.

To assess this finding we estimate the exponent of the distribution tail applying the *maximum likelihood estimator*, introduced by Hill (1975), conditional on the threshold x_{min}, thta is the *location* parameter (that is, the minimum value of individual wealth from which the tail starts).[18] Accordingly, the *shape* parameter of the Pareto distribution for values $\omega \geq \omega_{min}$ is given by:

$$\hat{\alpha} = \frac{1}{z}\left[\sum_{j=1}^{z}(\ln \omega_{n-j+1} - \ln \omega_{min})\right]^{-1}$$

where z is the number of observations above the threshold ω_{min}, and n is the total number of observations. For each ω_{min} in the wealth distribution we estimate the shape parameter $\hat{\alpha}$ and then compute the Kolmogorov-Smironov (K-S) goodness-of-fit test, obtaining a certain value of the statistic D by confronting the actual distribution with a theoretical one with the same parameters.[19] At the end, we choose that couple $(x_{min}, \hat{\alpha})$ for which we observe the minimum value of the statistic D and the K-S test does not reject the null hypothesis that the two distributions are drawn from the same underlying population. In the case of the baseline model we obtain a shape parameter $\hat{\alpha} = 2.2206$ corresponding to a location parameter $\omega_{min} = 4252.8$.[20]

Agents spend a different time in a role or in another: a large fraction of the population is part of the working class for the majority of simulation time; the average time spent by individuals in the capitalist role is slightly higher than 5%; in fact, few agents exhibit a time spent as a capitalist larger than 1/3 of the time (as shown in Figure 1, upper-right

[18] The method is based on minimising the "distance" between the theoretical power law distribution and actual data. For more details, see Clauset et al. (2009).

[19] As x_{min} increases, the number of observations above this threshold decreases. We proceed with the described procedure until z is larger than 1/10 of the entire sample in terms of observations, in order to apply the estimation procedure on a "too small" sample. Moreover, the random sample for the theoretical power law distribution is generated by inverse transform sampling: given the random sample U drawn from a uniform distribution in the interval [0,1], the variate $T = \omega_{min}U^{-(1/\alpha)}$ is Pareto distributed.

[20] The number of the observations involved in the estimation procedure leading to these parameter values is 362. In the total of the 900 iterations of the procedure, the K-S test did not reject the null hypothesis in the 56.78%, the majority of which for estimates based on a relatively "large" value of ω_{min}. Moreover, $\hat{\alpha} = 2.2206$ is in the range $1.5 - 2.5$ in which we can find many empirical estimates of the Pareto distribution for income and wealth for different countries (Nirei and Souma, 2007).

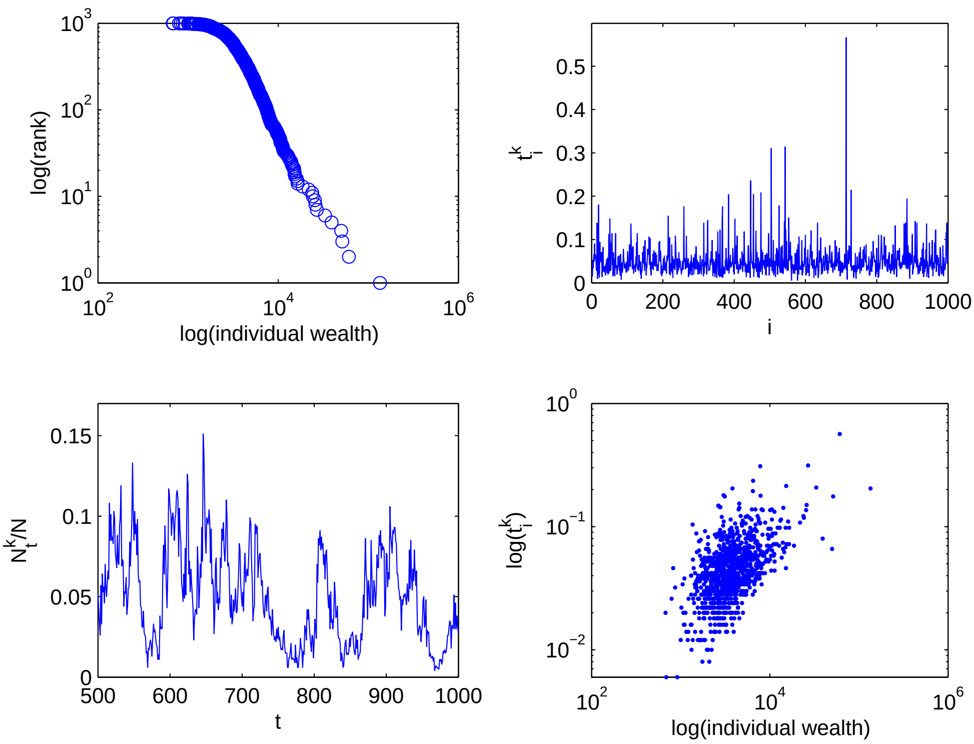

Figure 1: Wealth distribution and class division: *upper-left panel*, rank-size plot of individual wealth; *upper-right panel*, fraction of time each agent has spent as a capitalist; *bottom-left panel*: fraction of agents belonging to the capitalist class as time elapses; *lower-right panel*: logarithm of individual wealth vs. logarithm of time spent as capitalist.

panel); accordingly, the size of the capitalist class oscillates around a value slightly higher than 5% of the population (Figure 1, bottom-left panel).

A relationship between class division and wealth distribution clearly emerges: the longer the period of time spent as capitalist the higher the amount of individual wealth (see bottom-right panel of Figure 1). Then, for the agents in this economy, the aim of accumulating wealth is better reached being capitalists rather than workers; at the same time, the accumulation of wealth leads to a higher probability to be capitalists, and so on. This amplification mechanisms results in an unequal distribution of wealth which is connected to class division.

The correlation between class division and wealth distribution suggests that the agents which spend the longest period of time in the capitalist role are those who own the largest wealth, while agents who are workers for a long time tend to be poor. But the shape of the distribution also signals that there are "intermediate cases" for which the agents' wealth is not-so-small because of a certain period of time spent as capitalist. In other words, in the long run, all agents benefit from the stochastic returns of the multiplicative process although only a few for a considerable period of time. Some agents benefit from the multiplicative process of wealth accumulation enough time to be considered as "quasi-capitalist" or "working rich", while we could consider "working poor" the agents which staying, for a very long period of time in the working class, remain poor.

All in all, even in our simplified context in which an agent can play the role of capitalist or worker in a certain period of time, the possibility of switching between the roles – depending on wealth distribution dynamics – leads, in a sense, to the emergence of a *middle class* which benefits from the multiplicative stochastic process of wealth accumulation for a non negligeable time.[21] This result is related to the degree of *social mobility*, as measured by the fraction of people changing their role during the simulation, that in this case is almost 10%.

Finally, the profit rate oscillates around the mean of the multiplicative process for profits, that is $\mu_\phi = 0.05$, the actual value being 0.0422. This positive average profitability leads to an accumulation of wealth at an average growth rate of 0.0024 (henceforth, $g\bar{\Omega}$), with a standard deviation of 0.0111 (henceforth, $std\bar{\Omega}$).

4.3 Robustness check

Let's now assess the robustness of the results coming from the simulation of the baseline model. In order to do this, we perform a Monte Carlo experiment consisting of 100 simulations for different seeds of the pseudo-random number generation process. We collect statistics for the following variables: the average growth rate of wealth accumulation and its standard deviation (henceforth, $g\bar{\Omega}$ and $std\bar{\Omega}$, respectively); the Gini index ($gini$); the average fraction of time the agents spend in the capitalist role ($\%\tilde{t}^k$); the average percentage of agents changing their role during the simulation ($\%change$); the average value of the shape parameter of the wealth distribution tail ($\hat{\alpha}$) and the corresponding location parameter from which the tail starts (ω_{min}); the number of observations involved in the estimation procedure ($\#obs$); and finally, the average percentage of iterations for which the K-S test does not reject the null hypothesis of a power law tail ($\%KS$). For each variable we also compute the stardard deviation across Monte Carlo simulations.

The results of this computational experiment are summarised in Table 1 (*avg* stays for the Monte Carlo averages, *std* for the corresponding standard deviations) and show that model findings are quite robust to changes due to stochastic elements (that is, the pseudo-random number generation process). In particular, the wealth accumulation process grows at a 0.003 rate (with a 0.02 standard deviation across the 100 simulations). Wealth inequality as measured by the Gini index is around 0.38 (with a 0.03 st. dev.). Agents play the role of capitalists for an average time slightly higher than 4% (this is also

[21] In our context the status of 'working rich' or 'working poor' depends only on the time spent in the capitalist role, given that we assumed that agents are homogeneous in their abilities. An interesting direction to improve the model would be to consider different individual productivities leading to a high- vs. low-skill workers scenario.

	$g\bar{\Omega}$	$std\bar{\Omega}$	$gini$	$\%\bar{t}^k$	$\%change$	$\hat{\alpha}$	ω_{min}	$\#obs$	$\%KS$
avg	0.0030	0.0204	0.3839	4.21%	9.27%	2.1182	4035.4	426.75	55.44%
std	0.0012	0.0176	0.0299	0.0092	0.0105	0.2182	620.03	95.826	0.0501

Table 1: Robustness check.

the approximative average size of the capitalist class). Moreover, in the average the 9% of the population switch from a class to another during the simulation. As said above, this is a proxy for *social mobility* (together with the average size of the capitalist class).

Let's now briefly comment the result about the estimation of the wealth distribution parameters: the average value of the power law tail shape parameter is around 2.12 with a corresponding location parameter equal approximatively to more than 4000 (as the minimum level of wealth from which the Pareto distribution is estimated). An average number of observations is involved in the right tail of the wealth distribution equal to more than 400. Finally, the K-S test did not reject the null hypothesis that simulated data and theorethical data are drawn from the same underlying population; this result holds for more than 1/2 of the cases for which the estimation procedure was applied (clearly, the likelihood of this increases as ω_{min} increases).

4.4 Propensity to consume/save

In this subsection we analise the role of the parameters that control agents' propensity to consume their income and/or wealth, that is c', c'', and c'''. Results are summarised in Table 2 (where we also report the values regarding the Monte Carlo experiment on the baseline scenario with the label bm).

Following Kalecki (1942), we consider a first scenario (the computational experiment labelled as Ic) when the consumption/saving behaviour is strictly related to class division, assuming that $c' = 1$ and $c'' = 0$ – that is, workers spend all the labour income, while capitalists save all the profit income (while the value of c''' is the same as in the baseline scenario). In this case, the average growth rate of wealth accumulation is higher than in the baseline scenario. Also the volatility of the wealth accumulation process is significantly higher. Moreover, wealth inequality is "very high" ($gini = 0.8$). The shape parameter of the Pareto distribution of wealth is in this case equal to 1.18 and it is related to a very persistent class structure of the society, given that the average size of the capitalist class is "very small" and social mobility is "very low" (only the 2% of the population, in the average, change the status, from capitalist to worker or viceversa, during the simulation).

We consider, then, another scenario where both workers and capitalists have a "high" propensity to consume. We assume that all agents have the same propensity to consume equal to $c' = c'' = 0.7$ (this experiment is labelled as IIc). In this case, the lack of a large flow of savings results in a scarse accumulation of both individual and aggregate wealth. In fact, the growth rate of the wealth accumulation process is one/half of that in the baseline scenario. The "low" accumulation at the individual level leads then to a context in which a stable capitalist class does not emerge, because some agents do not become significantly richer than others and they do not develop the power to accumulate more and more wealth by playing the capitalist role in a systematic way. Accordingly, the Gini index is "low" given that agents spend one/third of the simulation in the capitalist role and continuosly change their status (almost 40% in the average). The K-S test signals that only in the 30% of the cases a power law tail emerges. This means that in the majority of the cases the maximum likelihood estimation procedure does not find a Pareto distribution. Only when the same procedure is applied to a very small sample of observations in the right tail of distribution the K-S test does not reject the null hypothesis of a power law shape. However, in these cases, the shape parameter is higher than 4. All in all, in this scenario agents are more equal in terms of wealth distribution but they are also poorer than in the baseline scenario.

A third scenario ($IIIc$) is based on a "low" propensity to consume for both workers and capitalists. In this case, we assume that $c' = c'' = 0.3$. Despite the agents' propensity to save is higher in this case than in the baseline scenario, in this setting the growth rate of the wealth accumulation process is smaller. This is because of a larger flow of savings boosting capital accumulation due to a not very persistent class division. Accordingly, wealth inequality is obviously lower as measured by the Gini index. The shape parameter of wealth distribution tail is almost equal to 2.3. The average time the agents play the capitalist role is almost the 7% of total time and it is related to an average rate of changing the status of almost 14%. This means that, due to a "low" propensity to consume, even agents playing the worker role have the possibility to accumulate a certain wealth, so contrasting the emerging power coming from capitalists' wealth accumulation. In other words, a well-established capitalist class emerges but, at the same time, the turnover between classes is not-so-low (13.8%) and a non-negligeable number of agents have the possibility to accumulate wealth by playing the capitalist role for enough time. As a conclusion, greater equality partially damages the "powerful" process of wealth accumulation which emerged with lower social mobility in our two classes economy.

In the fourth scenario (IVc) we investigate the role of c''', the propensity to consume wealth, setting a "high" value, that is $c''' = 1$. What emerges is similar to the IIc scenario in which both workers and capitalists have a "high" propensity to consume. However, a relevant difference is that in this case the growth rate of wealth accumulation is significantly higher. Another difference is that in this case the Gini index is slightly greater. The greater wealth inequality boosts accumulation both at the individual and the aggregate level, so confirming that class division leads to a powerful process of wealth accumulation.[22]

[22] An increasing level of *minimum consumption* $\bar{c}$ has an effect similar to that of a higher value of c''':

	$g\bar{\Omega}$	$std\bar{\Omega}$	$gini$	$\%\tilde{t}^k$	$\%change$	$\hat{\alpha}$	ω_{min}	$\#obs$	$\%KS$
bm	0.0030	0.0204	0.3839	4.21%	9.27%	2.1182	4035.4	426.75	55.44%
	(0.0012)	(0.0176)	(0.0299)	(0.0092)	(0.0105)	(0.2182)	(620.03)	(95.826)	(0.0501)
Ic	0.0141	0.1407	0.8067	0.47%	2.16%	1.1884	50.0746	388.57	49.87%
	(0.0122)	(0.0564)	(0.0960)	(0.0034)	(0.0050)	(0.1328)	(13.010)	(81.385)	(0.0489)
IIc	0.0015	0.0056	0.2216	29.72%	39.3%	4.7340	636.46	267.19	30.04%
	(0.0002)	(0.0003)	(0.0018)	(0.0103)	(0.0049)	(0.6551)	(76.194)	(80.654)	(0.0415)
$IIIc$	0.0024	0.0084	0.3431	6.76%	13.8%	2.2804	10898	405.18	51.21%
	(0.0004)	(0.0048)	(0.0074)	(0.0094)	(0.0103)	(0.2421)	(1842.3)	(98.035)	(0.0479)
IVc	0.0036	0.0156	0.2881	25.14%	38.03%	4.0287	83.8416	258.16	29.07%
	(0.0003)	(0.0007)	(0.0048)	(0.0150)	(0.0110)	(0.5639)	(12.939)	(77.595)	(0.0438)
Vc	0.0027	0.0180	0.3919	4.21%	9.05%	2.0979	3127.8	416.46	52.19%
	(0.0008)	(0.0127)	(0.0182)	(0.0082)	(0.0089)	(0.2225)	(505.96)	(92.9243)	(0.0488)

Table 2: Scenario analysis, propensity to consume/save.

The fifth scenario (Vc) considers a *relative minimum consumption* instead of a time-invariant level as above. In other words, we assume that the minimum consumption is now a fraction of average consumption: $\bar{c}_t = \theta c^m_{t-1}$, where c^m_{t-1} is the arithmetic mean of the consumption variable, and θ is a positive parameter. In fact, it has to be checked if agents have enough money to finance consumption. In the absence of capital markets, as it is assumed in our simplified model, credit is not allowed and the consumption level is financially constrained by agents' wealth. Accordingly, the minimum level of consumption for the agent i at time t is: $min\{\bar{c}_t, \omega^i_{t-1}\}$. In the proposed experiment θ is set to 0.25. There are no significant changes with respect to the baseline scenario but for a slightly higher Gini index, a slightly lower social mobility, and a slightly lower average value of the Pareto exponent.

Obviously, simulation results are sensitive to assumptions on the consumption/saving propensity. In general, a low propensity to consume – a high propensity to save – boosts wealth accumulation, and viceversa. This is a *classical political economy* result. The clearer case is when $c' = 1$ and $c'' = 0$, leading to the more unequal society we analised, because of a consumption behaviour stritly related to class division: capitalists save and workers consume. Interestingly enough, even when the propensity to save is high, independently of the role played in the society – the case for which $c' = c'' = 0.3$ – an evident class structure emerges, although characterised by a less unequal distribution of wealth. This is because of a greater competition among agents to become a capitalist which results in more social mobility and hampers the accumulation of power due to the growth of individual wealth relative to that of others. Instead, when $c' = c'' = 0.7$ the society evolves towards a quite egualitarian configuration but this also leads to a lower accumulation of wealth. For sure, the analysis suffers from a lack of investigation about the role of consumption and, in general, of the demand side of the economy. As we will discuss in the conclusions, this is one of the steps we are going to make in the next future

it decreases wealth accumulation due to a smaller flow of savings. For "high" values of $\bar{c}$ the Pareto tail of wealth distribution tends to disappear while wealth becomes quite equally distributed but significantly smaller both individually and socially.

	$g\bar{\Omega}$	$std\bar{\Omega}$	$gini$	$\%\bar{t}^k$	$\%change$	$\hat{\alpha}$	ω_{min}	$\#obs$	$\%KS$
bm	0.0030	0.0204	0.3839	4.21%	9.27%	2.1182	4035.4	426.75	55.44%
	(0.0012)	(0.0176)	(0.0299)	(0.0092)	(0.0105)	(0.2182)	(620.03)	(95.826)	(0.0501)
$low\gamma$	0.0027	0.0033	0.2954	7.89%	17.6%	2.6526	4667.9	399.58	52.29%
	(0.0033)	(0.0014)	(0.0091)	(0.0172)	(0.0170)	(0.3430)	(669.59)	(97.9627)	(0.0518)
$high\gamma$	0.0034	0.0383	0.4379	3.47%	6.94%	1.9111	3282.5	416.18	53.37%
	(0.0022)	(0.0250)	(0.0262)	(0.0055)	(0.0067)	(0.1595)	(455.08)	(73.0259)	(0.0547)

Table 3: Scenario analysis, the role of γ.

in order to improve the model.

4.5 The fraction of capital invested

In this subsection we investigate the role of γ in shaping simulation results. This parameter represents the fraction of wealth invested by a capitalist in the risky process described above. In a sense, then, it is a proxy for the riskiness of wealth accumulation, but we will also suggest a different interpretation. The parameter was set equal to 1/2 in the baseline model. Let's now check the behaviour of the simulation model for two different values of this parameter: $\gamma = 1/4$ and $\gamma = 3/4$, say a "low" and a "high" level of γ. For each of the two values we perform 100 simulations and summarise the results in Table 3, in which the experiments are labelled as $low\gamma$ and $high\gamma$, respectively.

For $\gamma = 1/4$ ($low\gamma$) the average growth rate of the wealth accumulation process is slightly smaller (and also less volatile) than that emerging from the simulation of the baseline model. Instead, wealth is more equally distributed as a consequence of a higher social mobility and an average time spent as capitalist of about 8%. The shape parameter of the power law tail is around 2.65 (then greater than the 2.12 of the baseline scenario).

For $\gamma = 3/4$ ($high\gamma$) wealth grows at a rate that is slightly larger than in the baseline scenario (and more volatile). The "high" value of the Gini index signals a quite unequal society. Accordingly, the average time agents play the capitalist role is smaller than 3.5%, with only the 7% of the population switching the role during the simulation time. The Pareto exponent of wealth distribution is around 1.9 in this case.

All in all, varying the value of γ especially impacts wealth inequality and, then, the interplay between wealth distribution dynamics and class division, rather than the average growth of the economic system (in terms of wealth accumulation). In a sense, the parameter γ could be interpreted as the amount of wealth required to perform the role of capitalist in the society. Accordingly, the larger γ the more likely is to observe an unequal society.

Trying to make a parallel with Galor (2006), a possible interpretation is that when economic growth is led by huge investments in *physical capital* (assuming that in our framework this leads to a large fraction of wealth invested in the risky process, that is a "high" γ), then a significant degree of inequality associated with a persistent class division emerges. On the contrary, if the access to the capitalist role is quite likely for many agents, that is to say when economic growth is led by investments in *human capital* (that could require a smaller amount of wealth to generate profits, that is a "low" γ in our framework), then a less unequal society emerges.[23]

Anyway, in both cases credit and finance would play a fundamental role, suggesting another way to improve our modelling framework. We can just say that the proposed random process of class selection implicitly means that agents' wealth is a sort of "collateral" in order to have the access to the capitalist role. Again, the lack of an appropriate analysis of the demand side of the economy – which for sure we expect to have a significant impact on macroeconomic dynamics – restricts the scope of the model findings. The experiments on the parameter γ suggest that the larger the fraction of wealth invested in the stochastic multiplicative process the more unequal is wealth distribution and the stronger are the effects of class division.

4.6 The parameters of the income generation processes

Agents obtain a stochastic income – wage or profit – depending on the random shock ϵ, that is a normally distributed variable with mean μ_ϵ and standard deviation σ_ϵ. Now we propose four additional experiments to see what happens when we change the values of these parameters. *Ceteris paribus*, we study model findings with $\mu_\epsilon = 0.01$ ($low\mu$), $\mu_\epsilon = 0.1$ ($high\mu$), $\sigma_\epsilon = 1/4$ ($low\sigma$), and $\sigma_\epsilon = 3/4$ ($high\sigma$). The results are summarised in Table 4. Figure 2 includes the rank-size plot of wealth distribution for the four proposed scenarios.

For a "low" μ_ϵ, the growth rate of the wealth accumulation process is smaller than in the baseline scenario, and aggregate volatility is lower. The shape parameter of the Pareto tail is a bit higher; moreover, the maximum likelihood estimation procedure works successfully only in about the 40% of the cases. In this scenario social mobility is slightly higher than in that of the baseline case and, consequently, the average time the agents play the capitalist role is higher (almost 7%). However, this does not lead to an egalitarian society given that wealth inequality, as measured by the Gini index, is similar to that of the baseline scenario. Qualitatively, the relevant difference is that this scenario is not characterised by a well-established Pareto tail of wealth distribution (see Figure 2, panel (a)).

For a "high" μ_ϵ, the accumulation of wealth proceeds at a higher growth rate and with more volatility. In this case wealth inequality is significantly greater than in the baseline

[23] Some words of caution are in order about the role of γ. It is clear that in our abstract framework we do not really distinguish between physical and human capital, as well as between high-skill and low-skill workers (as said above). Nevertheless, we think that even in a very stylised way this parameter provides a tool to investigate the complex interaction between the process of wealth accumulation and the related evolution of the social structure.

scenario ($gini = 0.55$) and this is related to "low" social mobility and a restricted class of agents playing the capitalist role. But, the estimation procedure leads to an average value of the shape parameter larger than 3. Moreover, the corresponding standard deviation across Monte Carlo simulations signals a significant dispersion around the mean (the same holds for $\%KS$). In fact, in this case wealth distribution clearly tends to be characterised by two regimes, one for the low to middle-wealth range and the other for the high-wealth range, with a "kink" in between, as shown in Figure 2, panel (b).[24]

For a "low" σ_ϵ, the wealth accumulation process grows at a rate slightly greater than in the baseline scenario. Despite the lower volatility at the individual level due to a "low" level of σ_ϵ, that is the random shock in the income generation processes, aggregate volatility (that is $std\Omega$) is slightly larger in this case than in the baseline one. This seems to be related to the emergence of a stable class division with a retricted capitalist class ($\%t^k$ smaller than 2%) although, at the same time, social mobility is not so lower with respect to the baseline scenario (almost 8% instead of more than 9%). Even though wealth is quite unequally distributed, as suggested by a Gini index almost equal to 0.4, the average value of the shape parameter tends to be almost 5. Moreover, the estimation procedure is successful only in about the 40% of the cases. Even in this case, we provide a rank-size plot of wealth distribution – Figure 2, panel (c) – that shows a different shape with respect to that emerging from the simulation of the baseline scenario.

For a "high" σ_ϵ, the growth rate of wealth accumulation is slightly lower than in the baseline scenario, while aggregate volatility is slightly greater. Wealth inequality is higher ($gini = 0.43$) while social mobility and the average time agents play the capitalist role are somewhat lower than baseline scenario values. Accordingly, in this case it emerges a Pareto tail of wealth distribution with a shape parameter around 1.93 (the estimation procedure was successfully applied in almost 50% of cases), as shown in Figure 2, panel (d).

To sum up, a two regimes wealth distribution with a "kink" in between emerges when μ_ϵ is "high". Something similar also emerges in the first of the four scenarios. Then, the parameter μ_ϵ seems to be responsible for the kink in the distribution. The only scenario for which a well-established power law tail emerges, according to the proposed maximum likelihood estimation procedure, is the last one with a "high" σ_ϵ. In this case the shape parameter of the Pareto distribution of wealth is lower than 2 and it is related to a persistent class division.

Finally, the smaller fraction of cases – with respect to the baseline model results – for which the estimation procedure does not reject the null hypothesis of a power law tail seems to suggest that the parameters of the income generation processes have to be

[24] This result is reminiscent of that proposed by Dragulescu and Yakovenko (2001), and Nirei and Souma (2007) – an exponential decay due to labour income dynamics and a power law tail due to capital returns – that we have discussed above.

	$g\bar{\Omega}$	$std\bar{\Omega}$	$gini$	$\%\tilde{t}^k$	$\%change$	$\hat{\alpha}$	ω_{min}	$\#obs$	$\%KS$
bm	0.0030	0.0204	0.3839	4.21%	9.27%	2.1182	4035.4	426.75	55.44%
	(0.0012)	(0.0176)	(0.0299)	(0.0092)	(0.0105)	(0.2182)	(620.03)	(95.826)	(0.0501)
$low\mu$	0.0017	0.0081	0.3862	6.73%	12.95%	2.1940	1001.2	342.74	40.88%
	(0.0003)	(0.0032)	(0.0067)	(0.0100)	(0.0120)	(0.2184)	(219.74)	(82.069)	(0.0460)
$high\mu$	0.0115	0.1006	0.5545	1.42%	4.29%	3.3141	9906.1	491.99	49.66%
	(0.0096)	(0.0618)	(0.1918)	(0.0094)	(0.0116)	(1.5093)	(1056)	(104.40)	(0.1921)
$low\sigma$	0.0051	0.0318	0.3966	1.90%	7.81%	4.7601	5281.4	434.34	42.30%
	(0.0034)	(0.0264)	(0.1693)	(0.0166)	(0.0235)	(1.6158)	(368.74)	(101.05)	(0.1740)
$high\sigma$	0.0026	0.0266	0.4320	4.15%	8.27%	1.9301	2901.1	396.69	48.49%
	(0.0010)	(0.0161)	(0.0157)	(0.0062)	(0.0071)	(0.1729)	(531.25)	(77.47)	(0.0480)

Table 4: Scenario analysis, the role of μ and σ.

set according to a certain balance between values. In other words, combining a "high" volatility of the random shock ϵ with a "low" mean would lead to unrealistic configurations of the wealth distribution. To a certain extent this holds for all the parameters and their numerous potential combinations. Accordingly, a possible way to further improve our framework would be to validate it through confronting simulation data and empirical ones, so calibrating the values of parameters.

4.7 Other computational experiments

In this Section we propose some additional experiments to further test the system behaviour whith different initial conditions in terms of the wealth/wage ratio, or when a positive return on the money not invested in the stochastic production process is introduced, or under an altenative parameter setting regarding the surplus division between workers' wages and capitalists' profits (as a proxy of social conflict).

Initial conditions: wealth/wage ratio. In Table 5 we report the results of various experiments perfomed with different initial conditions. Data reported show that simulation results are robust (in the long run) to changes in initial conditions. All in all, there are no significant differences across the alternative scenarios due to the initial level of the wealth/wage ratio (set to 100/1 in the *bm* case) but for a "scale effect" (see the values of ω_{min} in Table 5).

Positive return on non-invested capital. Table 6 refers to simulations in which we introduced a positive return on workers' saving and on the fraction of capital, $(1-\gamma)$, that capitalists do not invest in the stochastic production process. The slight increase of the growth rate and the parallel slight increase of inequality that emerge by setting the interest rate equal to 0.1% is clearly confirmed by simulations for which $int = 1\%$; in the latter case the Gini index tends to 0.5 and the power law scale parameter is around 1.6. Accordingly, a positive return on savings (considered as the money not invested in the stochastic process) reinforces the mechanisms leading to capital accumulation and a persistent class division (qualitative results are similar for higher values).

Social conflict. In this case, given the level of $\mu = 5\%$, we assume that the mean of

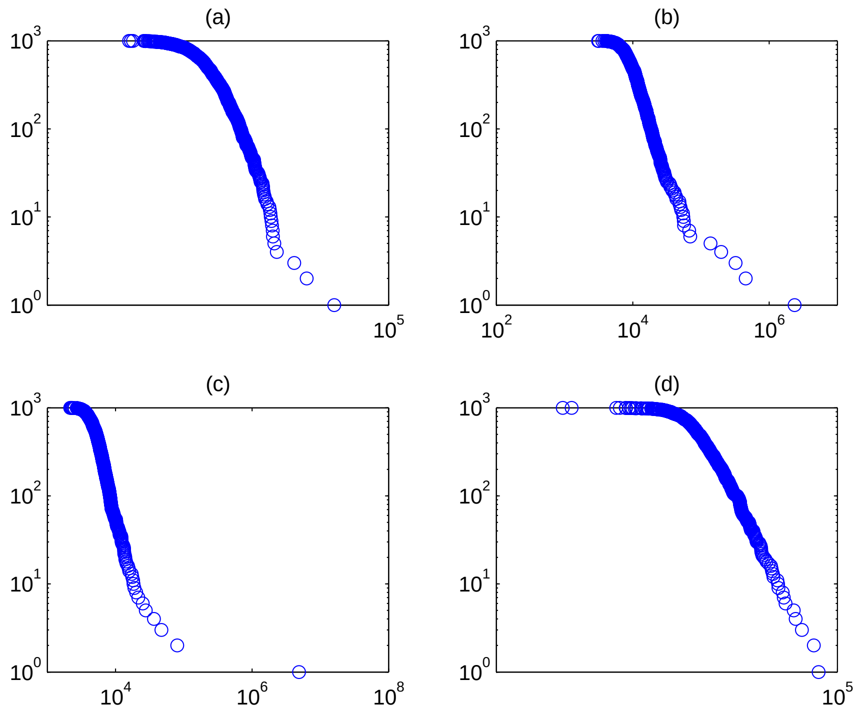

Figure 2: Rank-size plot of wealth distribution for (a) $\mu_\epsilon = 0.01$; (b) $\mu_\epsilon = 0.1$; (c) $\sigma_\epsilon = 1/4$; (d) $\sigma_\epsilon = 3/4$.

the wage stochastic process is given by $\mu + sc$ while the mean of the profit stochastic process is given by $\mu - sc$, where the positive parameter sc is a proxy for social conflict. Results are reported in Table 7. When sc is "high", the system grows at a lower pace and the fraction of social surplus to be appropriated by capitalists is "low", and viceversa. At the same time, there is more social mobility between classes and the inequality is lower (with a lower level of the Gini index and a higher value of the power law scale parameter). Instead, lower levels of sc lead to faster accumulation but also to lower mobility and higher inequality (the growth rate is equal to that of the baseline model when $sc = -0.01$, while it is higher with $sc = -0.03$). In the last scenario, however, the number of cases for which the KS test does not reject the hypothesis of a power law for wealth distribution is below 50% and the number of observations included in the distribution tail when estimating the model is below 400. Then, very few people are in the power law right tail when sc is "very low". All in all, from this simple computational experiment on the role of social conflict a trade-off between growth and equality emerges. Clearly, an explicit treatment of aggregate demand in a truly macroeconomic setting (one of the first modifications we are going to introduce in an improved version of our model) may strongly affect this basic result.

	$g\bar{\Omega}$	$std\bar{\Omega}$	$gini$	$\%\tilde{t}^k$	$\%change$	$\hat{\alpha}$	ω_{min}	$\#obs$	$\%KS$
bm	0.0030	0.0204	0.3839	4.21%	9.27%	2.1182	4035.4	426.75	55.44%
	(0.0012)	(0.0176)	(0.099)	(0.0092)	(0.0105)	(0.2182)	(620.03)	(95.826)	(0.0501)
$\omega_0 = 1000$,	0.0021	0.0188	0.3904	3.90%	7.62%	2.1037	4599.8	431.44	55.12%
$w_0 = 1$	(0.0006)	(0.0125)	(0.0198)	(0.0079)	(0.0096)	(0.2485)	(738.16)	(99.175)	(0.0510)
$\omega_0 = 10$,	0.0030	0.0186	0.3787	4.26%	10.5%	2.1494	4021.3	428.37	54.81%
$w_0 = 1$	(0.0009)	(0.0161)	(0.0294)	(0.0102)	(0.0111)	(0.3388)	(674.58)	(99.713)	(0.0579)
$\omega_0 = 1$,	0.0029	0.0190	0.3802	4.37%	11.27%	2.1115	3928.4	439.68	55.19%
$w_0 = 1$	(0.0009)	(0.0163)	(0.0278)	(0.0098)	(0.0112)	(0.2702)	(602.82)	(93.514)	(0.0578)
$\omega_0 = 100$,	0.0020	0.0212	0.3785	3.98%	10.31%	2.0341	16319	489.26	60.02%
$w_0 = 100$	(0.0012)	(0.0168)	(0.0235)	(0.0094)	(0.0105)	(0.2255)	(2529.4)	(92.170)	(0.0649)
$\omega_0 = 100$,	0.0018	0.0215	0.3850	3.94%	10.58%	1.9982	133130	485.07	60.88%
$w_0 = 1000$	(0.0015)	(0.0206)	(0.0350)	(0.0090)	(0.0106)	(0.2310)	(21600)	(98.437)	(0.0646)

Table 5: Scenario analysis, Initial conditions (wealth/wage ratio)

	$g\bar{\Omega}$	$std\bar{\Omega}$	$gini$	$\%\tilde{t}^k$	$\%change$	$\hat{\alpha}$	ω_{min}	$\#obs$	$\%KS$
bm	0.0030	0.0204	0.3839	4.21%	9.27%	2.1182	4035.4	426.75	55.44%
	(0.0012)	(0.0176)	(0.099)	(0.0092)	(0.0105)	(0.2182)	(620.03)	(95.826)	(0.0501)
$int = 0.1\%$	0.0032	0.0191	0.3932	4.03%	8.78%	2.0379	5624.2	438.76	55.11%
	(0.0014)	(0.0162)	(0.0345)	(0.0085)	(0.0094)	(0.2773)	(916.04)	(97.515)	(0.0513)
$int = 1\%$	0.0096	0.0199	0.4798	3.31%	7.69%	1.6082	1875600	383.65	48.70%
	(0.0009)	(0.0121)	(0.0207)	(0.0065)	(0.0083)	(0.1515)	(533410)	(99.277)	(0.0533)

Table 6: Scenario analysis, Positive return on non-invested capital

	$g\bar{\Omega}$	$std\bar{\Omega}$	$gini$	$\%\tilde{t}^k$	$\%change$	$\hat{\alpha}$	ω_{min}	$\#obs$	$\%KS$
bm	0.0030	0.0204	0.3839	4.21%	9.27%	2.1182	4035.4	426.75	55.44%
$(sc = 0)$	(0.0012)	(0.0176)	(0.0299)	(0.0092)	(0.0105)	(0.2182)	(620.03)	(95.826)	(0.0501)
$sc = 0.01$	0.0027	0.0138	0.3596	5.19%	10.85%	2.1455	4187.4	436.81	55.70%
	(0.0006)	(0.0106)	(0.0154)	(0.0086)	(0.0102)	(0.2252)	(670.01)	(94.788)	(0.0468)
$sc = 0.03$	0.0024	0.0094	0.3285	7.04%	13.77%	2.3131	4298.2	416.71	53.34%
	(0.0004)	(0.0073)	(0.0082)	(0.0108)	(0.0106)	(0.2596)	(769.51)	(91.81)	(0.0571)
$sc = -0.01$	0.0030	0.0237	0.4083	3.55%	7.90%	2.0527	3779.6	420.73	54.46%
	(0.0013)	(0.0194)	(0.0358)	(0.0076)	(0.0099)	(0.2021)	(635.99)	(103.05)	(0.0546)
$sc = -0.03$	0.0045	0.0487	0.4932	2.10%	5.30%	2.0532	2959.2	371.18	47.66%
	(0.0040)	(0.0350)	(0.0789)	(0.0072)	(0.0100)	(0.3556)	(456.25)	(76.74)	(0.0674)

Table 7: Scenario analysis, Social conflict

5 Conclusions

We have presented a simple stochastic model of wealth accumulation with class division. A major finding is that an unequal distribution of wealth may emerge starting from perfect equality, even in a society in which agents are homogeneous in their abilities. According to a *classical political economy* perspective, we analysed the interplay between the social structure of the economy and the wealth accumulation process. In the proposed framework, class selection is based on a random mechanism related to wealth distribution dynamics. In this setting, when a persistent class structure emerges, associated to low social mobility, wealth distribution evolves towards a Pareto or power law distribution that we estimated by applying a maximum likelihood method, checking the robustness of model findings through Monte Carlo simulations.

In the baseline model, we assumed that capitalists have a "high" propensity to save, while workers have a "high" propensity to consume. In general, playing the role of capitalist rises the probability of accumulate more and more wealth that, in turn, increases the probability to play again the role of the capitalist in the following period. This gives rise to an "amplification mechanism" leading to a persistent division in social classes according to which agents which spend a long time as a capitalist are more likely to become richer, while a large majority of the population remains in the working class, being characterised by a small probability to access the capitalist role. In this context, a power law characterises the "fat" right tail of wealth distribution.

As maintained by Levy (2003), the Pareto distribution for the high-wealth range seems to be due to *chance* rather than *differential abilities*. As a consequence, a very unequal distribution of wealth may emerge even when agents are homogeneous with respect to their talents. It is worth to remember that the result proposed by Levy (2003) refers to capital returns due to financial investments. Beyond agents' financial investment talent, however, in the economic reality different individuals are characterised by heterogeneous skills with a relevant impact on labour market and firm dynamics, as well as on macroeconomic performances, as the literature on the topic has largely shown. According to our vision, however, heterogeneous skills are mainly the result of differences in the family background, education, social environment, and so on, rather than of *innate abilities*. Moreover, we think that this assumption is in line with the Adam Smith's idea that the apparent difference in natural talents between people is a result of specialisation, not a cause: "The difference between the most dissimilar characters, between a philosopher and a common street porter, for example, seems to arise not so much from nature, as from habit, custom, and education"(Smith, 1776, Book I, Chapter 2, par. I.2.4.). Then, heterogeneous skills significantly depend on individual, familiar and social conditions that, in turn, are related to wealth accumulation and distribution.

In a capitalist system, in which the working of the economy is characterised by class division, individuals have different opportunities to accumulate wealth – and then to improve their skills, so their wealth, and so on – depending on the role played in the society: in our simple framework, while capitalists have access to a multiplicative stochastic process of wealth accumulation, an additive process characterises workers. On this basis, the model we proposed suggests that, in a society composed of agents which are homogeneous in talents and are initially endowed with an equal amount of wealth, a restricted capitalist class may appropriate a large fraction of wealth, so further reinforcing its *power* to accumulate even more wealth, while the working class remains relatively poor. Accordingly, we think that *chance* may have a role, but also that the *power relation* implicit in the class division characterising capitalist societies has a significant impact on the shape and the location of the power law distribution of wealth. In a sense, then, our stochastic model of wealth accumulation with class division suggests that the *power law* is the consequence of a *law of power*.

Exploring the model with alternative parameter settings highlighted some relevant properties. A "Kaleckian scenario" in which capitalists save and workers consume led to a very unequal society in which only very few agents benefit from the multiplicative process of wealth accumulation, so becoming much richer than the rest of the population. In this case, the shape parameter of the Pareto distribution of wealth is significantly smaller than in the baseline scenario, approaching the unity. A very different scenario emerges when agents have the same propensity to consume, independently of class division. This mitigates wealth inequality, especially when all agents have a "high" propensity to consume. However, in this case also the growth of aggregate wealth is significantly smaller than in the baseline model. Instead, wealth accumulation is stronger (but also more volatile) when all agents have a "high" propensity to save. This is associated to a greater wealth inequality than in the previous case. As a consequence, even agents playing the worker role have the possibility to accumulate a certain wealth, so contrasting the emerging power coming from the wealth accumulation of other agents: even if a quite well-established capitalist class emerges, at the same time the turnover between classes is not-so-low. A "high" propensity to consume a fraction of wealth led to conclusions similar to that of the scenario with a "high" propensity to consume profit/labour incomes.

Another relevant result is that the larger the fraction of wealth invested in the risky process the more unequal the society. This suggests that when the amount of wealth required to access the capitalist role is "large" – for instance, when economic growth is boosted by physical capital accumulation (if this may be paralled with a "high" γ in our framework) – an unequal society with a persistent class division emerges. By contrast, when this amount is "small" – for instance, when economic growth is boosted by human capital accumulation (that is, a "low" γ) – a less unequal society emerges. Hence, agents' wealth can be interpreted as a "collateral" in order to access the capitalist role.

The experiments for different values of the income generation processes' parameters showed that a two regimes wealth distribution with a "kink" in between may emerge, in particular when the mean of stochastic process is "high". The only scenario for which a well-established power law tail emerges is the one with a "high" volatility of the stochastic processes. In this case the shape parameter of the Pareto distribution of wealth is lower than 2 and it is related to a persistent class division. The smaller fraction of cases – with respect to the baseline model results – for which the estimation procedure does not reject the null hypothesis of a power law tail seems to suggest that the parameters of the income generation processes have to be set according to a certain balance between values. To a certain extent this holds for all parameters and their numerous potential combinations. Accordingly, a possible way to further improve our framework would be to validate simulation results with empirical data, so calibrating the values of parameters.

Other computational experiments show that a positive return on savings (defined as the wealth not invested in the stochastic process of production) reinforces the mechanisms leading to capital accumulation and a persistent class division. Moreover, different divisions of the surplus (due to social conflict) influence the dynamics of the model and the effects can be summarised as a trade-off between growth and equality (clearly, this result may change once aggregate demand is introduced in the model). Finally, simulation results are robust to changes in initial conditions: setting different levels of the wealth-to-wage ratio at time 0 do not significantly alter model findings in the long run (but for a scale effect).

The proposed framework suffers from some limitations which restrict the scope of results. For instance, we have not modelled the demand side of the economy that, for sure, plays a fundamental role in the reality (or, at least, we think so). We will move along this direction to improve the model by linking the stochastic process of class selection to the evolution of the profit rate,[25] the stochastic multiplicative process of profit generation to aggregate demand, and wages to capital accumulation. Another relevant aspect to introduce is credit as an external source to finance the wealth accumulation process, possibly amplifying wealth inequality due to asymmetric information. Furthermore, we could also introduce financial investments which allow agents to directly make financial profits (that is, outside the "real sector"). Finally, various policy experiments could be tested in an enriched setting. All in all, we think that the modelling of the demand side and of credit/finance will allow us to develop a stochastic *macroeconomic* framework to analyse the working of a capitalist system and its political dimension. However, the model we proposed is just a preliminary step towards a complex quest.

[25] We proposed such a mechanism in a preliminary version of the current work (Russo, 2011).

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