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Portfolio Frontiers with Restrictions to Tracking Error Volatility and Value at Risk

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Abstract

Asset managers are often given the task of restricting their activity by keeping both the value at risk (VaR) and the tracking error volatility (TEV) under control. However, these constraints can not always be simultaneously satisfied because the VaR is independent of the benchmark portfolio. The management of these restrictions is likely to affect portfolio performances and produces a wide variety of scenarios in the risk-return space. The aim of this paper is to analyse various interactions between portfolio frontiers when restrictions upon TEV and VaR are jointly imposed. Analytical solutions for the intersections are provided and short numerical methods are proposed when solutions are not available. Finally, a new portfolio frontier is introduced.

JEL Class.: C61, G11
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Portfolio Frontiers with Restrictions to Tracking Error Volatility and Value at Risk*

Giulio Palomba *Luca Riccetti*

1 Introduction

It is well known that investors assign part of their funds to asset managers who are given the task of beating a benchmark and the risk management usually imposes a maximum value on the tracking error volatility (TEV) in order to keep the portfolio risk close to that of a selected benchmark. Starting from the seminal contribution of Markowitz (1959) in the risk-return space (σ_P, μ_P), a lot of attention has been dedicated to constrained asset allocation strategies: for example, Jagannathan and Ma (2003) have provided evidence explaining why constraints are useful, while others, such as Boyle and Tian (2007), have studied the topic of outperforming a benchmark in the presence of constraints.[1] As defined by Franks (1992), the TEV is the most commonly used constraint of relative risk and is associated with the investment goal expressed in terms of the excess return over a benchmark; Roll (1992) showed that asset managers that aim to produce positive return performance over a benchmark whilst keeping TEV to a minimum usually select portfolios that are not mean/variance efficient.

Various asset allocation strategies have been proposed in the literature. The first is provided by Roll (1992), who suggests to restrict the portfolio's beta. Jorion (2003) shows that a TEV constraint produces an elliptic portfolio frontier in variance-return space,[2] whereas Alexander and Baptista

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[1] Various topics have been studied in the field of benchmarking, such as how to optimise costs in passive management by selecting the optimal number of assets to use (Jansen and Van Dijk, 2002) or by deciding when to rebalance (Gaivoronski, Krylov and Van der Wijst, 2005). Whether an active manager can beat the benchmark using a specific division of labour (Lee, 2000a) or other specific strategies (Browne, 1999) is another key topic in benchmarking and is used in the evaluation of the asset manager; see, for instance, Clarke, De Silva and Thorley (2002), Cremers and Petajisto (2009), Grinold and Kahn (2000) or Lo (2008).

[2] Various works use this methodology: for example El-Hassan and Kofman (2003) add further constraints such as no short-selling, Palomba (2008) inserts portfolio frontiers into an econometric model for asset allocation and Riccetti (2010) generalises the model of Jorion (2003) by inserting portfolio commissions.

(2008) impose a value at risk (VaR) constraint upon the standard asset allocation framework in (σ_P, μ_P) space. Alexander and Baptista (2010) recently presented a strategy of active portfolio management in which they use a target upon *ex ante* alpha, defined as the intercept when the portfolio return is linearly regressed on the benchmark return; in this case, the portfolio frontier contains all portfolios which minimise the TEV for any given *ex-ante* portfolio alpha.

The existence of various frontiers indicates the possibility of various asset allocation strategies. Hence, it would be interesting to identify one or more portfolio which is able to satisfy different criteria at the same time. This paper compares different portfolio frontiers and provides a summary of their graphical and analytical properties. The field of investigation is restricted to the framework of the quadratic utility function, which is strictly related to the assumption of normally distributed expected returns and the theories about rational expectations and market efficiency. Short sales are also allowed. Several portfolios of interest are also calculated and discussed, focussing on those that lie on the intersections between the different frontiers.

The scope of this work is to analyse the situations in which managers have to keep both the VaR and TEV under control. In doing so, from the economic perspective managers have to face two problems: first, TEV constrained portfolios could not satisfy the VaR restriction and second, TEV-VaR constrained portfolios are usually inefficient because they lie at the right of the so-called "Mean-Variance Frontier" (hereafter MVF).

The remainder of the paper proceeds as follows: section 2 consists of a summary of the principal portfolio frontiers provided in the literature; particular attention is devoted to the frontiers introduced by Jorion (2003) and Alexander and Baptista (2008) whose possible intersections are successively discussed in section 3; some numerical methods for determining common portfolios are also presented in section 3. In section 4, a new boundary for which TEV and VaR constraints can be satisfied at the same time is introduced. Section 5 closes the analysis with a short empirical example and section 6 concludes. An Appendix containing some useful proofs and results is also provided.

2 Review of portfolio frontiers

Before introducing the portfolio frontiers, some notation is required: assuming that the available data consist of n assets, their expected returns are contained in an n-dimensional column vector μ, while the squared $n \times n$ matrix Ω represents the covariance matrix. In accordance with the literature, the following constants are defined: $a = \iota'\Omega^{-1}\iota$, $b = \iota'\Omega^{-1}\mu$, $c = \mu'\Omega^{-1}\mu$ and $d = c - b^2/a$, where ι is an n-dimensional column vector in which each element is 1. As all these parameters are derived exclusively by the data, they are independent of any allocation strategy. In this setup, some subjective inputs are also relevant because risk managers could impose some constraints upon asset managers activity: in particular, they could set a desired level of total return (μ_P) or put restrictions upon TEV (T_0) and/or VaR (V_0) into place. The geometric analysis will mostly be conducted in the (σ_P^2, μ_P) space, whereas all the graphical implications are shown in the usual (σ_P, μ_P) space in which the axes refer to the absolute risk and total return respectively.

Our study will be conducted taking two fundamental portfolio frontiers into account: the popular MVF, first introduced by Markowitz (1959), and the "Mean-TEV Frontier" (hereafter MTF) defined by Roll (1992). It is well known that the MVF consists of all the portfolios which minimise the total portfolio variance, given a desired portfolio return; its equation is

$$\sigma_P^2 = \sigma_C^2 + \frac{1}{d}(\mu_P - \mu_C)^2, \tag{1}$$

which produces a parabola in the (σ_P^2, μ_P) space or a hyperbola in the (σ_P, μ_P) space. The expected return $\mu_C = b/a$ and the variance $\sigma_C^2 = a^{-1}$ are of the minimum variance portfolio (portfolio C), which is independent of the desired portfolio return μ_P. All the portfolios for which $\mu_P \geq \mu_C$ belong to the efficient subset of MVF.

The MTF, on the other hand, shifts the asset allocation strategies from the absolute risk perspective to that of the risk relative to a benchmark portfolio $B \equiv (\sigma_B^2, \mu_B)$; in this context, the TEV becomes the risk component that the manager aims to minimise instead of the total portfolio variance. The equation for the MTF is thus

$$\sigma_P^2 = \sigma_B^2 + \frac{1}{d}(\mu_P - \mu_B)^2 + 2\frac{\Delta_1}{d}(\mu_P - \mu_B), \tag{2}$$

where $\Delta_1 = \mu_B - \mu_C$. The constant Δ_1/d does not depend upon the expected portfolio return. Comparing equations (1) and (2), it is evident that the MTF is a horizontal translation of the MVF in the variance-mean space; hence, it is easy to show that these curves have no intersections. These frontiers have the same analytical form with the only exception of the third addend in equation (2) which contributes to the above mentioned translation. The MVF is calculated "around" the minimum variance portfolio (C), while the benchmark is the reference portfolio for the MTF, but does not correspond to its minimum.[3] However, Roll (1992) claims that portfolios belonging to the MTF are generally suboptimal because they lie to the right of the MVF and are thus overly risky. The horizontal distance between the frontiers in the (σ_P^2, μ_P) space represents the efficiency loss (δ_B): for each

[3] In (σ_P^2, μ_P) space, the minimum portfolio in equation (2) is $G \equiv (\sigma_B^2 - \Delta_1^2/d, \mu_C)$.

value of the portfolio return (μ_P), this distance is obtained by subtracting equation (1) from equation (2), hence

$$\delta_B = \Delta_2 - \frac{\Delta_1^2}{d}, \tag{3}$$

where $\Delta_2 = \sigma_B^2 - \sigma_C^2$. Given the impossibility of any intersection between MVF and MTF, δ_B is positive for all μ_P by construction. However, in the special case of the benchmark lying on the mean-variance boundary, a contact can occur: in such a context, the benchmark minimises both the portfolio variance and the TEV at the same time and the relationship $\Delta_2 = \Delta_1^2/d$ corresponds to equation (1) for $\mu_P = \mu_B$; in this situation, the efficiency loss is clearly zero.

2.1 The Constrained TEV Frontier (CTF)

Jorion (2003) adds the specific constraint TEV=T_0 to the Markowitz setup, thus obtaining the "Constrained TEV Frontier" (hereafter CTF), a closed and bounded frontier in the (σ_P^2, μ_P) space whose equation is

$$d(\sigma_P^2-\sigma_B^2-T_0)^2+4\Delta_2(\mu_P-\mu_B)^2-4\Delta_1(\sigma_P^2-\sigma_B^2-T_0)(\mu_P-\mu_B)-4d\delta_B T_0 = 0, \tag{4}$$

where Δ_1, Δ_2 and δ_B are as previously defined. Jorion (2003) shows that equation (4) is that of an ellipse for which the horizontal axis has a positive (negative) slope when $\Delta_1 > 0$ ($\Delta_1 < 0$). The horizontal centre of the ellipse is $\sigma_B^2 + T_0$, hence an increase in T_0 produces a surface area expansion. This elliptic frontier becomes somewhat distorted in the (σ_P, μ_P) space. Since we assume that asset managers generally face constrained optimisation, the CTF contains the benchmark and all the feasible portfolios for which TEV$\leq T_0$; as shown in Jorion (2003) and Palomba (2008), restricting TEV influences the ellipse eccentricity, thus it is important to have one or more intersections between the CTF and the MVF. Specifically, the number of contacts depends on the equation

$$\Psi = d\ T_0 - d\Delta_2 + \Delta_1^2. \tag{5}$$

When $\Psi < 0$ the frontiers do not intersect, but when $\Psi = 0$ they have one portfolio in common. When $\Psi > 0$, the frontiers have two intersections which tend to move along the MVF as the value of T_0 increases. These contacts define two arcs on the ellipse: the left arc coincides with the mean-variance boundary, while the right arc represents the TEV constrained frontier. The most interesting situation is $\Psi = 0$ for which the frontiers are tangent and the tangency TEV is

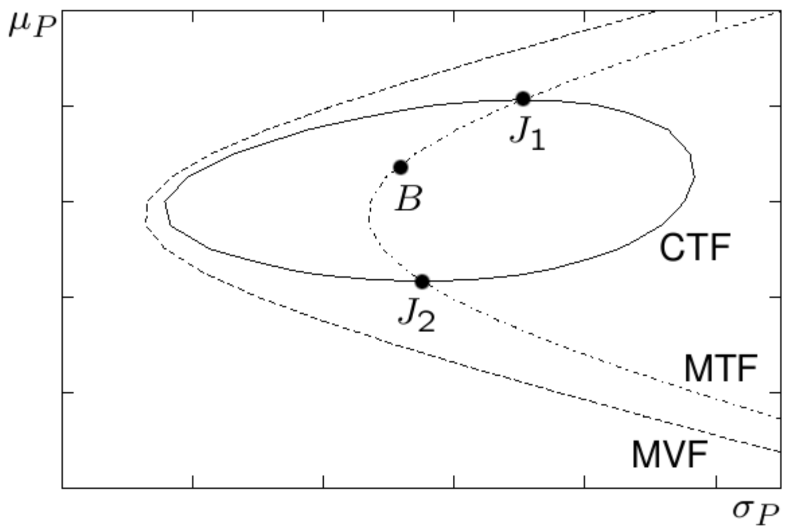
$$T_H = \delta_B = \Delta_2 - \frac{\Delta_1^2}{d}, \tag{6}$$

where $H \equiv (\sigma_C^2 + \Delta_1^2/d, \mu_B)$ is the contact between the frontiers.[4] Jorion (2003) also shows that the CTF intersects the MTF in portfolios

$$\begin{cases} J_1 \equiv (\sigma_B^2 + T_0 + 2\Delta_1\sqrt{T_0/d}, \mu_B + \sqrt{dT_0}) \\ J_2 \equiv (\sigma_B^2 + T_0 - 2\Delta_1\sqrt{T_0/d}, \mu_B - \sqrt{dT_0}), \end{cases} \tag{7}$$

corresponding to those portfolios with the maximum and minimum expected return respectively. The economic interpretation of this result is straightforward because the minimum TEV boundary coincides with the constrained TEV frontier for portfolios with TEV=T_0. Given that portfolios in equation (7) belong to the MTF, their efficiency loss is δ_B. Defining $J_1 \equiv (\sigma_1^2, \mu_1)$ and $J_2 \equiv (\sigma_2^2, \mu_2)$, where $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$, all the portfolio frontiers are shown in Figure 1: portfolio J_1 gives more return and more risk than J_2, therefore managers could maintain a TEV=T_0 by selecting a position in the so-called efficient CTF, defined as the left arc $\widehat{J_1 J_2}$; the efficiency loss of portfolios which lie on this arc is less than that of portfolios lyng on the arc formed by J_1 and J_2 on MTF.

Figure 1: Portfolio frontiers when $\Delta_1 > 0$



2.2 The constrained value at risk frontier (CVF)

Given the distribution of the portfolio return, the VaR is the maximum loss $V_0 = z_\theta \sigma_P - \mu_P$, where z_θ is the critical value taken from the standardised normal distribution with a confidence (significance) level $0.5 < \theta < 1$; the VaR=V_0 is fixed by risk managers and defines the intercept of the con-

[4] Moreover, equation (6) confirms that $\delta_B \geq 0$ for all μ_P.

strained VaR frontier (hereafter CVF),

$$\mu_P = z_\theta \sigma_P - V_0, \tag{8}$$

a linear frontier in the (σ_P, μ_P) space, where the slope is provided by z_θ which is always positive, while the intercept $(-V_0)$ should be negative.[5] This frontier is independent of the benchmark. The constraint imposed by the CVF is satisfied for positive σ_P in the space to the left of the straight line. Alexander (2009) has shown that if this constraint binds, the CVF surely intersects the MVF, determining a portfolio that is efficient by construction. Nevertheless, the intersection between MVF and CVF can be empty, thus several VaR constrained portfolios are not admissible: a natural consequence is that its intersections with MTF and CTF can also be empty. Using the asymptotic slope of MVF as the critical value, Alexander and Baptista (2008) distinguish a low confidence level ($0 < z_\theta \leq \sqrt{d}$) from a high confidence level ($z_\theta > \sqrt{d}$) and then provide a detailed discussion about the VaR constrained frontiers for different scenarios. Focussing on the slope and the intercept in equation (8), this type of analysis consists of a problem of analytical geometry problem: in this context, the objective is reaching intersections between the hyperbola MVF and a sheaf of straight lines depending upon parameters V_0 and z_θ. When this problem admits a solution, a "Constrained Mean-TEV Frontier" (hereafter CMTF) is defined in the (σ_P, μ_P) space: this is the frontier which satisfies the VaR constraint and it is composed of portfolios with the smallest TEV. According to this definition, the CMTF could be

(i) an empty set if the CVF does not intersect the MVF,

(ii) a single portfolio if the CVF is tangent to the MVF,

(iii) a segment if the CVF crosses the MVF only,

(iv) an arc consecutive to two segments if the CVF crosses both the MVF and the MTF (see Alexander and Baptista, 2008, for details).

3 Intersections between the CTF and the CVF

Searching for the intersections between the CTF and the CVF is especially interesting in practical situations in which risk managers are required to select a portfolio with restrictions on both TEV and VaR; this constrained strategy implies allocations which are generally suboptimal because they do

[5] Formally, the intercept of the CVF should be negative to represent a loss, while a positive value would indicate an earning. In practice, the relationship $-V_0 < 0$ is not guaranteed, thus the definition of VaR is extended here from "the maximum expected loss" to the "worst expected return".

not belong to the MVF. However, the objective is to determine a non-empty subspace within (σ_P, μ_P) in which risk managers could set a bound on VaR in the presence of restrictions on TEV. Given that the CVF is a straight line with a positive slope whose left half-plane contains all the portfolios with VaR$< V_0$, it can have zero, one or two contacts on the CTF, depending on T_0, V_0 and θ. However, the following problems arise:

1. if the CVF lies to the left of the CTF, then the VaR constraint is too stringent. In such a case an intersection does not exist and it is thus possible to satisfy one constraint at the most between TEV$=T_0$ and VaR$=V_0$;
2. if the CVF intersects the CTF, then at least one portfolio satisfies both restrictions upon TEV and VaR. Specifically, a unique solution exists when the CVF is tangent to the CTF on the left, whereas two contacts occur and an infinite number of solutions are available when the CVF crosses the CTF;
3. if the CVF lies to the right of the CTF, the VaR constraint is uneffective.

The first two (relevant) scenarios are illustrated in Figure 2 which focuses on the VaR constraint. A critical value V_K should exist for which the curves become tangent. In particular, only when $V_0 \geq V_K$ is there at least one contact between the frontiers; such intersections occur independent of the sign of Δ_1 and the positions of portfolios J_1 and J_2. From the analytic perspective, the intersections between the CTF and the CVF correspond to the solutions of a system that includes the ellipse in equation (4) and the parabola (8) in the (σ_P^2, μ_P) space. The resolvent of such a system is the quartic equation

$$c_1\mu_P^4 + c_2\mu_P^3 + c_3\mu_P^2 + c_4\mu_P + c_5 = 0 \tag{9}$$

where

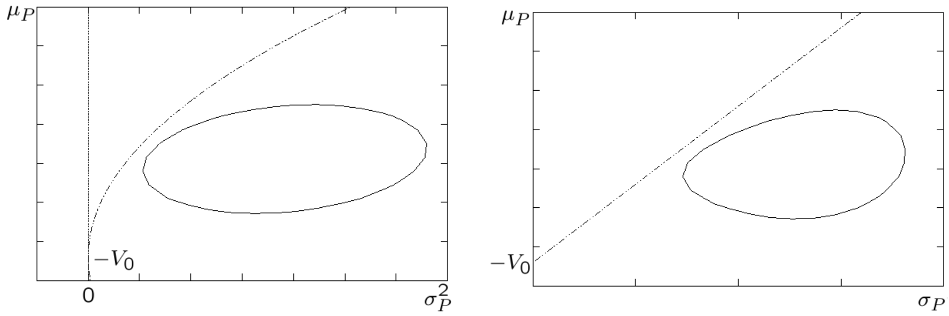
$$\begin{cases} c_1 = f(\theta) \\ c_2 = f(\theta, V_0, \mu_B) \\ c_3 = f(\theta, V_0, T_0, \mu_B, \sigma_B^2) \\ c_4 = f(\theta, V_0, T_0, \mu_B, \sigma_B^2) \\ c_5 = f(\theta, V_0, T_0, \mu_B, \sigma_B^2). \end{cases}$$

The analytical expressions for the coefficients are provided in the Appendix. These parameters determine the positions of the frontiers (see Figure 2), thus:

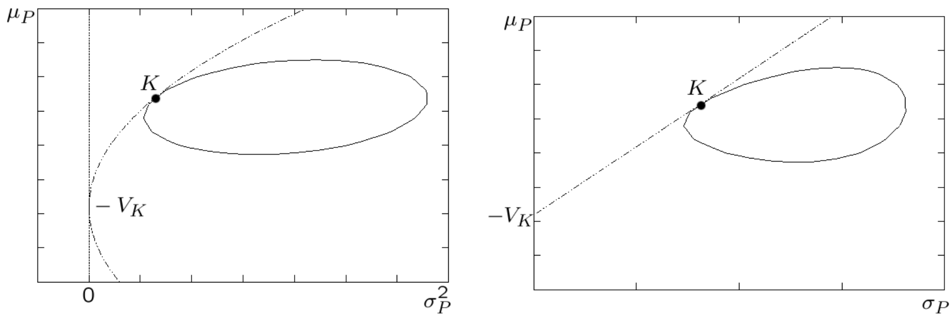
- θ influences the magnitude of the parabola,
- V_0 determines the position of the parabola's vertex along the μ_P-axis,

Figure 2: Contacts between CTF and CVF

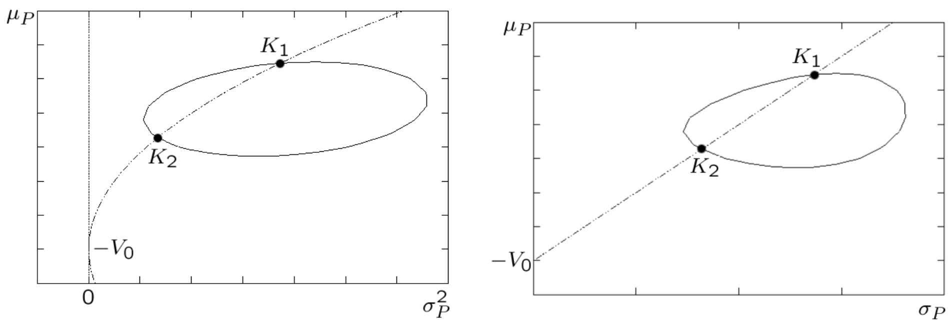
(a) Stringent constraint ($V_0 < V_K$)



(b) Tangency constraint ($V_0 = V_K$)



(c) Two intersections ($V_0 > V_K$)



- T_0 affects the eccentricity of the ellipse,
- the benchmark portfolio indicates the position of the ellipse.

3.1 Two contacts between CTF and CVF

Unfortunately, the algebra in solving equation (9) is messier than that of the other cases in which contacts between the frontiers are sought. Given this, it is not always possible to find analytical solutions for this polynomial and instead some numerical techniques are required.[6] Our proposed solution consists of minimising the function

$$S(\mu_P) = [S_T(\mu_P) - S_V(\mu_P)]^2, \tag{10}$$

where

$$\begin{cases} S_T(\mu_P) = \sigma_B^2 + T_0 + \dfrac{2}{d}\left\{\Delta_1(\mu_p - \mu_B) - \sqrt{d\delta_B[dT_0 - (\mu_p - \mu_B)^2]}\right\} \\ \\ S_V(\mu_P) = \left(\dfrac{\mu_P + V_0}{z_\theta}\right)^2. \end{cases} \tag{11}$$

Both equations express the portfolio variance: $S_V(\mu_P)$ is the correspondent of equation (8) in the (σ_P^2, μ_P) space, while $S_T(\mu_P)$, with $\mu_2 \leq \mu_P \leq \mu_1$ (see equation (7)), derives from equation (4), as documented in the Appendix. Clearly, if an intersection occurs, the minimum value of the function (10) should be zero.

The function $S(\mu_P)$ is differentiable and locally convex[7] around two minima (or zeros) and the BFGS algorithm represents a suitable and not computationally cumbersome instrument for obtaining the contacts between the CTF and the CVF. However, some characteristics of the entire optimisation process have to be discussed. First, the general solution is always provided by the contact portfolios $K_1 \equiv (\sigma_{K_1}^2, \mu_{K_1})$ and $K_2 \equiv (\sigma_{K_2}^2, \mu_{K_2})$; in this context, the starting value of μ_P in the optimisation is decisive for the convergence to minima. Second, when the frontiers do not intersect, function $S(\mu_P)$ always admits a unique minimum value that is different from zero. In this situation the algorithm provides the expected return at which the horizontal distance between the frontiers is minimised.

3.2 Tangency

Considering equation (10) to be a function of two variables $S(\mu_P, V_0)$, first order conditions are satisfied along the line $\mu_P = -V_0$.[8] This means that

[6] In particular, equation (9) should admit two real solutions (and two complex conjugates) when the portfolio frontiers intersect. See Appendix.

[7] The first order conditions are discussed in the Appendix.

[8] In short: $\dfrac{\partial S(\mu_P, V_0)}{\partial V_0} = -2[S(\mu_P, V_0) - S_V(\mu_P, V_0)]\dfrac{\mu_P + V_0}{z_\theta^2}$.

there is an infinite number of minima and that $S(\mu_P, V_0)$ is not convex with respect to V_0; hence, BFGS optimisation cannot be used for finding the minimal V_K at which the CTF and the CVF are tangent to each other. To address this problem, we propose the following grid search process

$$\begin{cases} \text{for} & \mu_i = \mu_1, \mu_1 - \varepsilon, \mu_1 - 2\varepsilon, \ldots, \mu_2 \\ \underset{\mu_i}{\text{Min}} & V_i = z_\theta S_T^{1/2}(\mu_i) - \mu_i \end{cases} \tag{12}$$

where $S_T(\mu_i)$ is as defined in equation (11), $\mu_2 < \mu_i < \mu_1$ and ε is an arbitrary and numerically small increment. The algorithm in equation (12) is fast to compute and returns a numerical solution whose numerical accuracy strictly depends upon the magnitude of the increments. The solution is $V_K = \min\{V_i\}$, which corresponds to the VaR constraint relative to the tangency portfolio K; the coordinates of such a portfolio are μ_K and $\sigma_K^2 = S_T(\mu_K)$.

4 The Fixed VaR-TEV Frontier (FVTF)

In this section we derive a new portfolio frontier. Before starting the analysis, a fundamental distinction has to be made about parameter Δ_1 because its sign is that of the horizontal axis of the ellipse in the (σ_P^2, μ_P) space, as documented in Jorion (2003).

4.1 Horizontal axis of the ellipse with positive slope

The assumption $\Delta_1 > 0$ means that the benchmark return is superior to that of the minimum variance portfolio (C) and that the horizontal axis of the CTF has a positive slope. When a VaR constraint is also imposed, it becomes binding, but only when the CVF intersects the efficient CTF; in this context, the slope z_θ becomes crucial in determining the relationship between VaRs V_1 and V_2 for which the straight line passes through J_1 and J_2 respectively (see Table A-4 for a summary).

4.1.1 High confidence level, no contacts between MVF and CTF

The analysis starts with the triple $\Delta_1 > 0$, $z_\theta > \sqrt{d}$, $T_0 < T_H$, indicating that the ellipse has a positive slope, that the confidence level is high (see Alexander and Baptista, 2008) and that the CTF does not intersect the MVF. In practice, a high confidence level is the most realistic, provided that the VaR constraint is generally imposed with a $\theta \geq 0.9$. However, we have substantially revised the scenarios introduced by Alexander and Baptista (2008) by also taking the TEV restriction into account; hence, analysis is carried out in order to identify the intersections lying between the portfolio frontiers and a parallel sheaf of lines. From the geometric perspective, the whole of the discussion is illustrated in Figure 3:

(a) small bound: the imposed VaR constraint is $V_0 < V_M$, where V_M is the VaR at which the CVF is tangent to the MVF. The analytical solution for V_M is

$$V_M = -\mu_C + \sqrt{\sigma_C^2(z_\theta^2 - d)} \tag{13}$$

corresponding, in (σ_P^2, μ_P) space, to portfolio

$$M \equiv \left(\sigma_C^2 + d\frac{\sigma_C^2}{z_\theta^2 - d}, \mu_C + d\frac{\sigma_C}{\sqrt{z_\theta^2 - d}} \right) \tag{14}$$

which only depends upon the confidence level θ (proof in the Appendix). Under the small bound condition, the imposed restriction is too stringent, the straight line CVF thus lies to the left of the MVF. There are no feasible portfolios which satisfy these restrictions upon both TEV and VaR.

(b) minimum bound: the position M, at which $V_0 = V_M$, provides the only admissible solution being the tangency portfolio between the MVF and the CVF. Nevertheless, this VaR restriction does not satisfy the restriction TEV$\leq T_0$.

(c) strong bound: this situation is only available when $V_M < V_0 < V_K$, where V_K is the value of VaR at which the CVF is tangent to the CTF. When this restriction holds, the CVF only intersects the MVF in portfolios $M_1 \equiv (\sigma_{M_1}^2, \mu_{M_1})$ and $M_2 \equiv (\sigma_{M_2}^2, \mu_{M_2})$ for which the analytical coordinates are provided in the Appendix. According to Alexander and Baptista (2008), the admissible solution is the closed and bounded space between arc $\widehat{M_1M_2}$ and the segment $\overline{M_1M_2}$; segment $\overline{M_1M_2}$ represents the CMTF. Nevertheless, the restriction on TEV cannot be satisfied in this portion of the (σ_P, μ_P) space.

(d) medium bound: in this case, $V_0 = V_K$, thus the CVF is tangent to the CTF in portfolio $K(\sigma_K^2, \mu_K)$ which allows asset managers to earn TEV$=T_0$. The region containing admissible portfolios is the same as in the previous case, but here portfolio $K \in \overline{M_1M_2}$ defines a new frontier: this is the "Fixed VaR-TEV Frontier" (FVTF), which includes all admissible portfolios which satisfy the VaR constraint and guarantee a TEV that does not exceed an *ex-ante* fixed value T_0.

(e) intermediate bound: when $V_K < V_0 < V_R$, the CVF crosses the MVF (portfolios M_1 and M_2) and the CTF (portfolios K_1 and K_2). The constraint on VaR is thus less stringent than in the previous case and CMTF is again provided by segment $\overline{M_1M_2}$, as observed by Alexander and Baptista

(2008). The value

$$V_R = -\mu_C + \sqrt{\left(\sigma_B^2 - \frac{\Delta_1^2}{d}\right)(z_\theta^2 - d)}, \tag{15}$$

where $V_R > V_M$, represents the VaR constraint at which the CVF is tangent to MTF (see Appendix). This tangency occurs in portfolio

$$R \equiv \left(\sigma_C^2 + \frac{d\sigma_B^2 - \Delta_1^2}{z_\theta^2 - d}, \mu_C + \sqrt{d\frac{d\sigma_B^2 - \Delta_1^2}{z_\theta^2 - d}}\right), \tag{16}$$

which is independent of the VaR constraint V_0. In such a situation, asset managers can satisfy both the VaR and the TEV restrictions within the closed and bounded FVTF, the subspace inside the CTF lying to the left of the CMTF. In Figure 3 (e), the FVTF corresponds to the arc $\widehat{K_1K_2}$ and the segment $\overline{K_1K_2}$, where $\mu_{K_1} > \mu_{K_2}$. For each $\mu_{K_2} < \mu_P < \mu_{K_1}$, asset managers have to make a choice: they can reduce the TEV below T_0 by augmenting the overall risk or they can maintain TEV$=T_0$ and consequently reduce the efficiency loss.

(f) maximum bound: when $V_0 = V_R$, the FVTF is identical to the previous case with the exception of portfolio R in which the TEV is minimised by definition. Hence, the VaR restriction binds all along the segment $\overline{M_1M_2}$, while in segment $\overline{K_1K_2}$ asset managers maintain TEV$\leq T_0$. On the other hand, all along the arc $\widehat{K_1K_2}$, managers can reduce the overall portfolio risk by maintaining a fixed TEV.

(g) large bound: in this situation $V_R < V_0 < \hat{V}$ with $\hat{V} = \max\{V_1, V_2\}$; thus, the CVF crosses the MVF in portfolios M_1 and M_2, the CTF in portfolios K_1 and K_2 and the MTF in portfolios R_1 and R_2. The FVTF is given by the arc $\widehat{K_1K_2}$ to the left of the CMTF ($\overline{K_1R_1}$, $\widehat{R_1R_2}$ and $\overline{R_2K_2}$). Furthermore, the portfolios lying within the segment $\overline{R_1R_2}$ do not belong to the FVTF since they are dominated by those in $\widehat{R_1R_2}$: on the one hand, the TEV in $\overline{R_1R_2}$ is not minimised and, on the other hand, it is possible to obtain the same TEV for portfolios in $\overline{R_1R_2}$ to the left of the MTF (arc $\widehat{R_1R_2}$), thus reducing the efficiency loss.

(h) larger bound: this bound occurs when $V_0 = \hat{V}$. In this case, the FVTF corresponds to the two arcs formed by portfolios J_1 and J_2 on the frontiers CTF and MTF. Portfolios lying within the CTF to the right of the arc $\widehat{J_1J_2}$ on the hyperbola MTF are all dominated portfolios: this is the reason why the restriction on $V_0 = \hat{V}$ is the largest that asset managers can impose.

Table A-4 shows that $V_1 < V_2$ when $\sqrt{d} < z_\theta < z_\theta^*$ and $V_2 < V_1$ when $z_\theta > z_\theta^*$, where the reference slope z_θ^* guarantees that the CVF passes through both J_1 and J_2; formally, this value depends on T_0 hence

$$z_\theta^* = \frac{d}{2\Delta_1}(\sigma_1 + \sigma_2). \tag{17}$$

For simplicity, the plot in Figure 3 (h) is obtained under the condition $z_\theta = z_\theta^*$ for which the following theorem applies.

Theorem 1 *In the (σ_P, μ_P) space, when $\Delta_1 > 0$, the straight line passing through portfolios J_1 and J_2 has a slope that is greater than the asymptotic slope of the MVF ($\sqrt{d}$).*

(proof in Appendix)

(i) no bound: when $V_0 > \hat{V}$, the VaR constraint does not operate and the FVTF is the same defined in the larger bound case.

4.1.2 High confidence level, one contact between the MVF and the CTF

As in the previous section $\Delta_1 > 0$ is set. Here, the imposition of $T_0 = T_H$ determines a unique intersection between the MVF and the CTF; hence, $T_0 = T_H$. The role of tangency portfolio $H \equiv (\sigma_C^2 + \Delta_1^2/d, \mu_B)$ is crucial in this analysis because it might also occur that the CVF is tangent to the MVF in H: this is a special case in which the FVTF is given by portfolio $M \equiv H \equiv K$, the minimum bound in Figure 3 (b) and the medium bound in Figure 3 (d) become the same VaR restriction and the strong bound in Figure 3 (c) cannot to be imposed because $V_M = V_K$. When $M \equiv H \equiv K$, the slope of CVF is

$$z_\theta^H = \sqrt{d + \frac{d^2\sigma_C^2}{\Delta_1^2}}, \tag{18}$$

where $z_\theta^H > \sqrt{d}$ by definition and

$$z_\theta^* > z_\theta^H. \tag{19}$$

The Appendix contains the proofs of equations (18) and (19). All the other scenarios with $V_0 > V_K = V_H = V_M$ remain identical to those illustrated in Figure 3. Furthermore, when $z_\theta = z_\theta^H$, equation (19) indicates that $\hat{V} = V_2$.

When $z_\theta \neq z_\theta^H$, and therefore $M \neq H \neq K$, two different scenarios could arise: if $\sqrt{d} < z_\theta < z_\theta^H$, it follows that $\mu_B < \mu_K < \mu_M$ while, if $z_\theta > z_\theta^H$, it follows that $\mu_M < \mu_K < \mu_B$, as shown in Figure 4. In both cases, $V_M < V_K < V_H$ and minimum, strong and medium bounds exist.

Figure 3: $\Delta_1 > 0$, high confidence level, $T_0 < T_H$

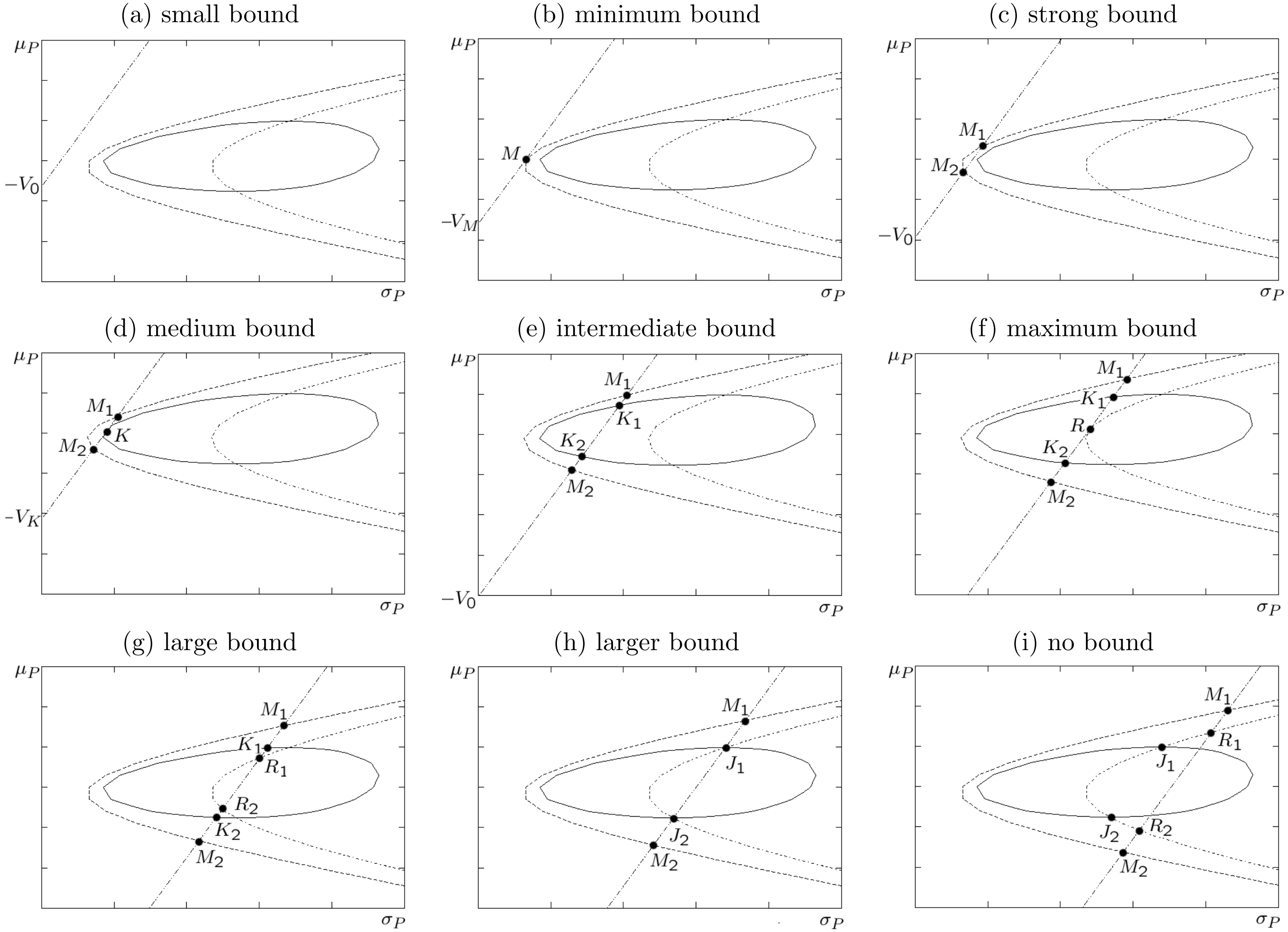
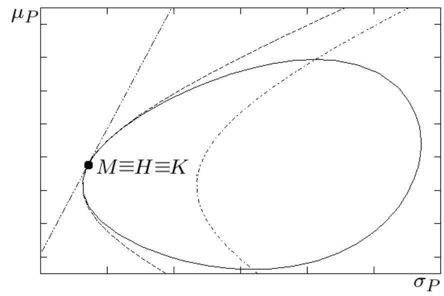
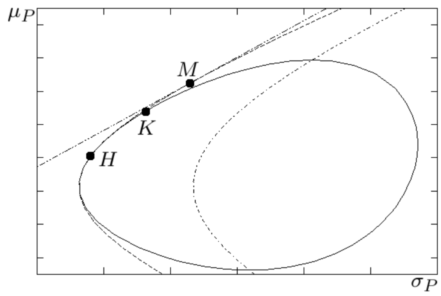


Figure 4: $\Delta_1 > 0$, high confidence level, $T_0 = T_H$. All Figures are plotted with the CVF passing through portfolio M

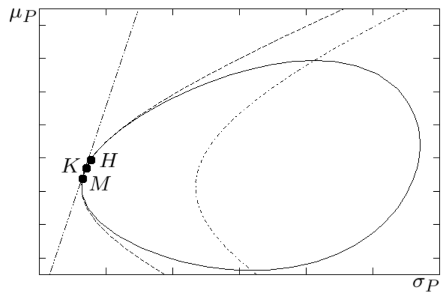
(a) $z_\theta = z_\theta^H$



(b) $\sqrt{d} < z_\theta < z_\theta^H$



(c) $z_\theta > z_\theta^H$



4.1.3 High confidence level, two contacts between the MVF and the CTF

When $T_0 > T_H$, the TEV constraint is feeble and the CTF intersects the MVF in two distinct portfolios, thus forming the arc $\widehat{H_1H_2}$ whose length augments when $\Psi > 0$ in equation (5) increases (see Palomba, 2008); in this context, portfolio $H \in \widehat{H_1H_2}$ by definition, $\mu_{H_2} < \mu_B < \mu_{H_1}$ and the FVTF is the same as defined in the previous sections. However, depending on z_θ, Ψ and V_0, each of the following relationships may occur: $\widehat{K_1K_2} \cap \widehat{H_1H_2} = \varnothing$, $\widehat{K_1K_2} \cap \widehat{H_1H_2} \neq \varnothing$, $\widehat{K_1K_2} \subset \widehat{H_1H_2}$ and $\widehat{H_1H_2} \subset \widehat{K_1K_2}$.

In practical situations, an interesting scenario emerges when the condition $M \in \widehat{H_1H_2}$ holds: in such a situation, the minimum VaR bound $V_0 = V_M$ is sufficient for obtaining a portfolio which satisfies both TEV and VaR restrictions. Conversely, when $M \notin \widehat{H_1H_2}$, the expected return of the tangency portfolio M could be greater than that of portfolio H_1 or less than that of portfolio H_2: in the former case, M lies on the MVF efficient set, to the right of H_1, where the tangency can only be reached for slopes z_θ that are close to the MVF asymptotic slope $\sqrt{d}$. In the latter case, the tangency may only occur when $\Psi > 0$ is sufficiently small to guarantee the condition $\mu_C < \mu_M < \mu_{H_2}$.

4.1.4 Low confidence level

From the analytical perspective, when a low confidence level ($z_\theta \leq \sqrt{d}$) applies, the CVF cannot be tangent to the two hyperbolic frontiers MVF and MTF in (σ_P, μ_P) space. The whole analysis is summarised by Figure 5, in which the condition $T_0 < T_H$ is adopted for simplicity.

(a) strong bound: as clearly shown in Alexander and Baptista (2008), an intersection always exists between the straight line CVF and the frontiers MVF and MTF (portfolios M and R).[9] When $V_0 < V_K$, asset managers have to make a choice between VaR and TEV because it is impossible to obtain V_0 and T_0 at the same time.

(b) medium bound: in this case $V_0 = V_K$ and the FVTF is given by K, which is the tangency portfolio between the CVF and the CTF: portfolio K represents the sole position at which manager can satisfy both VaR and TEV restrictions.

(c) intermediate bound: when $V_K < V_0 < V_1$ the CVF intersects the MTF outside the CTF, thus the FVTF is composed of $\overline{K_1K_2}$ and $\widehat{K_1K_2}$, where K_1 and K_2 are the contact portfolios belonging to both the CVF and the ellipse.

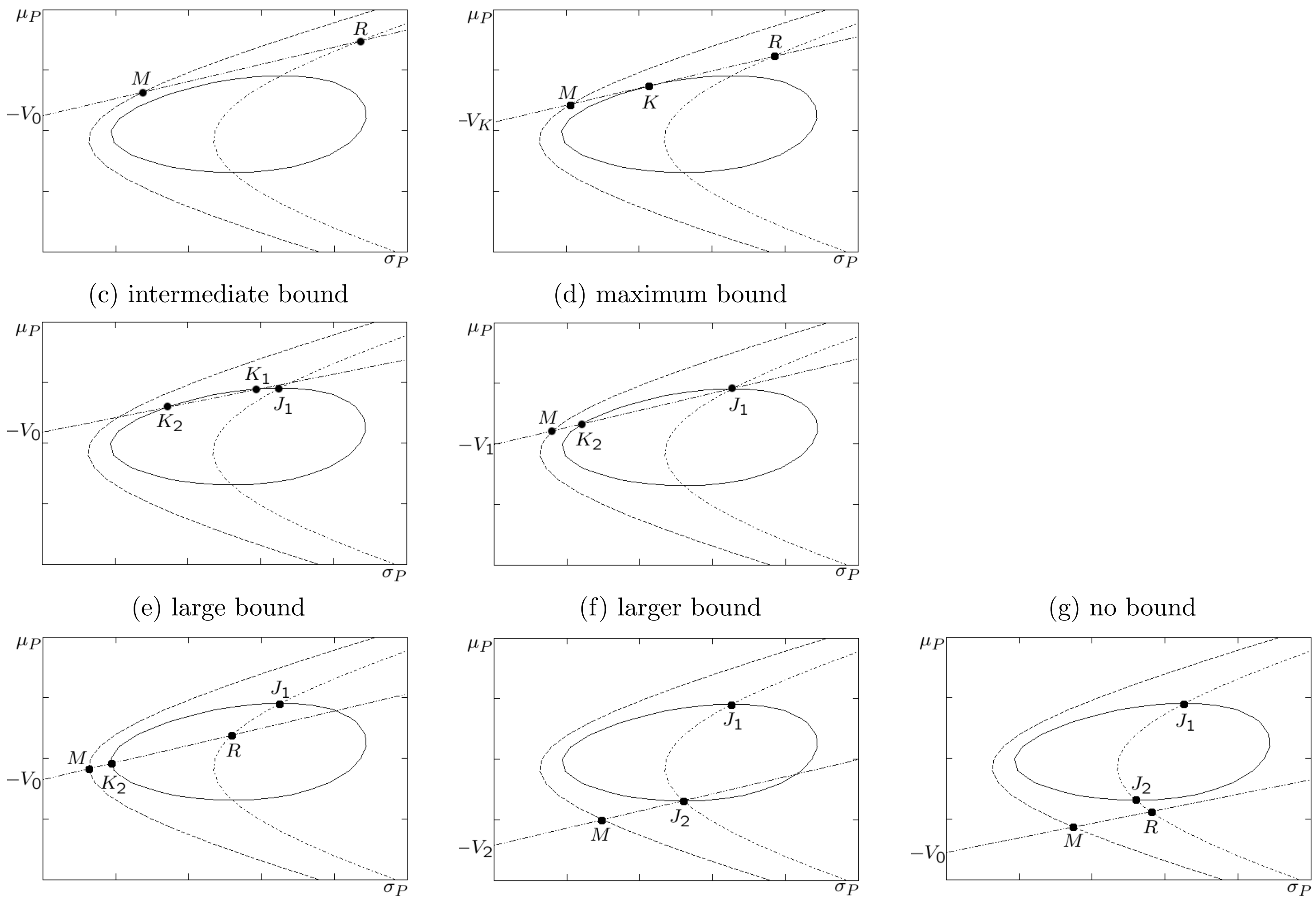
(d) maximum bound: "maximum" because it corresponds to the more stringent VaR restriction at which the FVTF has a portfolio in common with MTF: specifically, the bound $V_0 = V_1$ implies that the CVF passes through portfolio $R \equiv J_1$, thus FVTF is simply provided by the segment $\overline{K_2J_1}$ and arc $\widehat{K_2J_1}$.

(e) large bound: in such a situation $V_1 < V_0 < V_2$, where V_2 is defined as the VaR restriction in portfolio J_2; the FVTF is generally composed by arcs $\widehat{K_2J_1}$ and $\widehat{RJ_1}$ and segment $\overline{K_2R}$ that belongs to he straight line CVF. Portfolio R is the intersection between the MTF and the CVF.

(f) larger bound: when $V_0 = V_2$, the straight line CVF passes through portfolio J_2 and the portfolios composing the FVTF corresponds to arcs $\widehat{J_1J_2}$ belonging to both the MTF and the CTF (to the left of MTF).

[9] The slope $z_\theta = \sqrt{d}$ represents the only exception: Alexander and Baptista (2008) show that when $V_0 \leq -\mu_C$, the CVF does not intersect the MVF. Moreover, when $-\mu_C < V_0 \leq -\mu_C + \sqrt{d\delta_B}$, the CVF only intersects the MVF: in this case, the contact portfolio R does not exist.

Figure 5: $\Delta_1 > 0$, low confidence level $z_\theta < \sqrt{d}$, $T_0 < T_H$



(a) strong bound (b) medium bound

(c) intermediate bound (d) maximum bound

(e) large bound (f) larger bound (g) no bound

(g) no bound: when $V_0 > V_2$, the VaR constraint is uneffective and the FVTF is as described in the larger bound scenario.

When $T_0 \geq T_H$, all the above scenarios remain substantially unaltered and the analysis could therefore be extended to situations in which the MVF and the CTF do intersect.

4.2 Horizontal axis of the ellipse with non positive slope

When $\Delta_1 < 0$, the horizontal axis of the ellipse CTF has negative slope in (σ_P^2, μ_P) space, while it has zero slope when $\mu_B = \mu_C$. Under these assumptions, the scenarios plotted in Figures 3 and 5 are substantially confirmed as are the discussions of the previous sections. In such a situation, the relevant differences are:

(i) $\sigma_1 \leq \sigma_2$ and $\mu_1 > \mu_2$, thus no feasible VaR constraints pass through $J_1 \equiv (\sigma_1, \mu_1)$ and $J_2 \equiv (\sigma_2, \mu_2)$: in particular, the slope z_θ^* in equation (17) would be negative when $\mu_B < \mu_C$ or infinite when the ellipse in the (σ_P^2, μ_P) space has a horizontal axis with zero slope;

(ii) the relationship $V_1 < V_2$ applies for any $0.5 < \theta < 1$;

(iii) scenarios similar to those documented in Figure 4 are not available. Portfolio H lies on the inefficient arc of the MVF, thus it can not coincide with the tangency portfolio M.

4.3 Extreme benchmarks

In the previous sections we showed that portfolio R always provides the position at which asset managers can minimise TEV using the most stringent VaR constraint possible; its analytical coordinates (σ_R^2, μ_R) are provided by equation (A-9) in the Appendix. When the confidence level is high, portfolio R represents the tangency portfolio between the MTF and the CVF; according to equation (16), it is independent of V_0 and managers are only able to minimise the TEV when $V_0 \geq V_R$.

However, asset managers can opt for a very stringent TEV restriction and the low eccentricity of the ellipse in the (σ_P^2, μ_P) space can generate a scenario more complex than those presented in the previous sections. In particular, once T_R defines the TEV of portfolio R, the following condition could be violated: the tangency portfolio R could be placed outside the CTF, thus $T_0 < T_R$ and $\mu_R \notin [\mu_1, \mu_2]$. In such a situation, the maximum bound shifts from V_R to V_1 when $\Delta_1 \leq 0$, or to V_2 when $\Delta_1 > 0$. In both cases, these bounds are the most stringent VaR constraints that allow managers to select a portfolio on the MTF lying inside CTF.

Hence, given $z_\theta > \sqrt{d}$, we consider the benchmark portfolio B for which $T_0 < T_R$, as extreme; heuristically, an extreme benchmark is a benchmark that lie far from the position R. Moreover, when $\Delta_1 > 0$, the so-called aggressive benchmarks could belong to this category. Figure 6 shows various situations in which portfolio R lies outside the CTF and $\Delta_1 < 0$ (when $\Delta_1 > 0$ the scenarios are the same). Specifically, using V_R as the reference VaR constraint, Figures 6 (a)-(b)-(c) show the medium VaR bound in the presence of extreme benchmarks. In particular, Figures 6 (b)-(c) highlight that the relationship $V_K \geq V_R$ could occur and this implies that the straight line CVF in the medium bound scenario surely intersects the MTF. Figures 6 (d)-(e)-(f) illustrate the intermediate ($V_K < V_0 < V_1$), maximum ($V_0 = V_1$) and large ($V_1 < V_0 < V_2$) bounds respectively, when the benchmark is extreme.

When the confidence is low, a benchmark can not be extreme because asset managers are always able to minimise the TEV by simply setting a VaR constraint. More precisely, for each VaR bound, a straight line CVF crossing the MTF must exists, thus an infinite number of intersections is available; these contacts are not tangency portfolios and their coordinates strictly depend on V_0 (see equation (A-9) in the Appendix). As a consequence, it is not possible to select a unique reference portfolio R on the MTF.

5 An empirical example

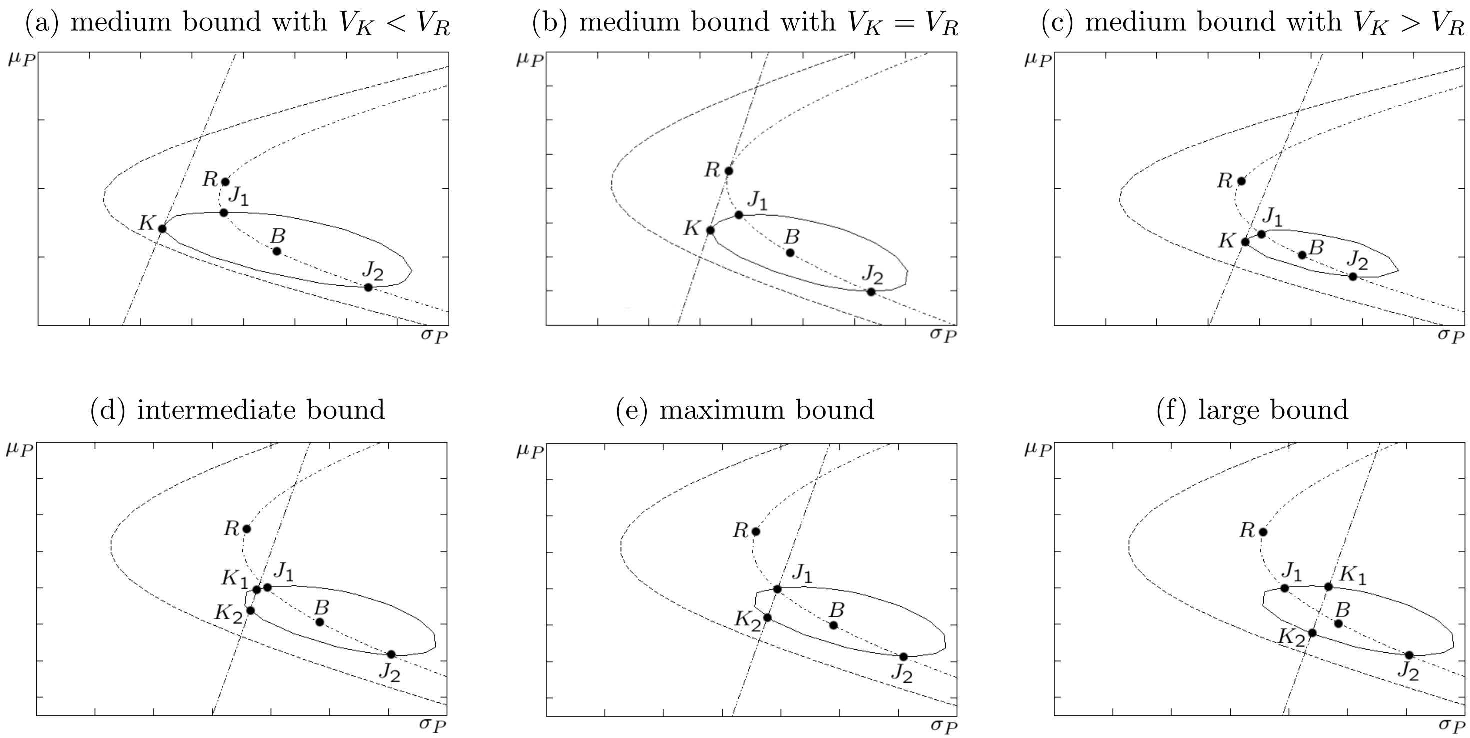
5.1 Data

In order to illustrate the theoretical results of section 4, a short empirical example is now provided which consists of an asset allocation problem among ten asset classes; both TEV and VaR constraints are taken into consideration. The available data are given by the quarterly returns (in percentages) of the 50 stocks composing the DJ Eurostoxx 50 index.

The data set runs from the first quarter of 2003 to the fourth quarter of 2010 and the sample size is 32.[10] The DJ Eurostoxx 50 and the Standard & Poor 500 Composite indices are used as benchmark portfolios (B). In line with Palomba (2008), all the stocks are grouped into 10 distinct asset classes, as shown in Table A-1.

[10]Data were sourced from Thomson DataStream. The prices for Alcatel and Crédit Agricole were unavailable prior the last quarter of 2001 are unavailable, thus the available dataset starts from the first quarter of 2002. We decided to use a restricted sample to avoid negative expected returns for the benchmark portfolios.

Figure 6: Extreme benchmarks ($T_0 < T_R$)



5.2 Results

The analysis is carried out by imposing an expected portfolio return $\mu_P = 5.00$ and setting the constraints $T_0 = 20.00$ and $V_0 = 15.00$. The results are provided in Tables A-2 and A-3: in the first case, the benchmark is the DJ Eurostoxx 50 index and Δ_1 is negative, while in the second case, the benchmark is the Standard & Poor 500 and Δ_1 is positive. In both cases a high confidence level of $\theta = 99\%$ and a low confidence level of $\theta = \Phi(\sqrt{d})$ are set. All the results are obtained for the following battery of portfolios:

- P is the portfolio lying on the MVF with an expected return of 5%,
- T is the portfolio lying on the MTF with an expected return of 5%,
- J is the portfolio lying on the CTF with an expected return of 5%,
- AB is the portfolio lying on the CMTF (see Alexander and Baptista, 2008) with an expected return of 5%,
- B is the benchmark,
- C is the minimum variance portfolio on the MVF,
- Q is the portfolio with the maximum Sharpe Ratio (Sharpe, 1994),
- J_1 and J_2 are the intersections between the MTF and the CTF,
- H is the portfolio on the MVF with the same return as the benchmark. It is also the contact portfolio between the MVF and the CTF when $\Psi = 0$,
- M and R are the tangency (intersection) portfolios between the CVF and the two hyperbolic frontiers when the confidence level is high (or low),
- K represents the tangency portfolio between the CTF and the CVF,
- K_2 is the left intersection between the CVF and the CTF.

For each of these portfolios the expected return, variance, risk, Sharpe Ratio, portfolio Alpha, TEV and Information Ratio (IR=Alpha/TEV, see for instance Lee, 2000b) are evaluated. Moreover, the efficiency loss and the intercept of the straight line CVF (the VaR restriction for a given θ) are also calculated. For the intersections M_1, M_2, K_1, K_2, R_1 and R_2, illustrated in Figure 3, only the coordinates within (σ_P, μ_P) space are provided.

When the confidence level is high, the scenario is as illustrated in Figure 3; in Table A-2 $\Delta_1 < 0$ determines the conditions discussed in section 4.2. In such a situation, the slopes z^*_θ and z^H_θ cannot be calculated. In Table A-3, portfolio J is not defined because the return of portfolio J_1 is less than

5%. The relationship $V_M < V_K < V_R < \hat{V}$, where $\hat{V} = V_2$, indicates that asset managers could impose any of the different restrictions upon VaR: small ($V_0 < V_M$), minimum ($V_0 = V_M$), strong ($V_M < V_0 < V_K$), medium ($V_0 = V_K$), intermediate ($V_K < V_0 < V_R$), maximum ($V_0 = V_R$), large ($V_R < V_0 < V_2$), larger ($V_0 = V_2$) or no bound ($V_0 > V_2$). In the specific case of a fixed $V_0 = 18.00$, the intermediate bound scenario occurs, hence the intersection portfolios M_1, M_2, K_1 and K_2 (see Figure 3 (e)) are also determined. In Table A-2 $\mu_R \notin [\mu_2, \mu_1]$, rendering the benchmark extreme ($T_R = 80.674$); on the other hand, in Table A-3 $T_R = 5.888$, thus the Standard & Poor index is not an extreme benchmark. In both Tables, an intermediate VaR bound is set and the FVTF is given by the segment $\overline{K_1K_2}$ and the arc $\widehat{K_1K_2}$ on the CTF.

When the confidence level is low, the scenario is that of Figure 5. Each of the portfolios P, T, J, B, C, Q, J_1, J_2 and H are independent of the confidence level θ, therefore they remain the same as those calculated for the high confidence level. In Table A-3, the constraint $V_0 = 5.00$ is set because the value of 15.00 is too feeble (it corresponds to the "no bound" situation) when the CVF has a small slope. In both Tables, portfolios M and R are the intersections between CVF and the hyperbolic frontiers the MVF and MTF respectively; K_2 represents the portfolio lying on the CVF with the maximum reduction of efficiency loss, while the VaR restrictions in J_1 and J_2 have been corrected for by taking the change in the confidence level into account. The coordinates of the tangency portfolio K differ from those calculated when the confidence level is high, while portfolio AB lies on the MTF ($\mu_P > \mu_M$), thus coinciding with portfolio T. However, the FVTF is composed by segment $\overline{RK_2}$ and arcs $\widehat{J_1K_2}$ and $\widehat{RJ_1}$.

6 Concluding remarks and further research

The key task faced by asset managers is that of beating a benchmark. Considering this and the fact that risk management usually involves measures to keep risks under control, this paper attempts to formalise asset allocation strategies in the presence of constraints put upon tracking error volatility (TEV) and value at risk (VaR); all the results are obtained considering the classical hypothesis of normally distributed expected returns which translates into an optimisation in the (σ_P^2, μ_P) space. However, Roll (1992) shows that portfolio optimisation based upon a relative risk measure is generally overly risky. Indeed, the so-called "Mean-TEV Frontier" (MTF), which is the set of portfolios with a given expected return and the smallest TEV, is usually far from the Markowitz efficient frontier ("Mean-Variance Frontier", MVF) with a portfolio's efficiency loss. Alexander and Baptista (2008) previously tried to reduce a portfolio's efficiency loss with the "Constrained Mean-TEV Frontier" (CMTF), which contains portfolios that satisfy a VaR constraint and have the smallest TEV.

This frontier does not take two economic problems into consideration. First, the VaR constraint is independent of the benchmark portfolio, hence it is not related to the maximum TEV constraint: in such a situation, restrictions on VaR and TEV cannot be satisfied at the same time. Indeed, Jorion (2003) highlights that asset managers can choose within a closed and bounded set of feasible portfolios ("Constrained TEV Frontier", CTF) that lie around the benchmark. The resulting asset allocation strategies thus suffer from a substantial reduction of the (σ_P, μ_P) space that is tightly connected to the restrictions imposed. Second, portfolios lying on the CMTF usually have a higher efficiency loss when compared to those lying to the left side of the CTF; this is due to the definition of the CMTF (see Alexander and Baptista, 2008) which is focussed on finding the smallest TEV. This paper shows that the imposition of a maximum and fixed TEV allows managers to move away from the CMTF and select less risky portfolios, thereby reducing efficiency loss. The above two problems can be summarised as follows: if maximum TEV and VaR limits are not compatible, there are no feasible portfolios and, at most, only one of the two constraints can be satisfied. Compatibility between maximum TEV and minimum VaR constraints only arises if the related portfolio frontiers intersect; in such a situation, or in the absence of a constraint on TEV, portfolios on the CMTF are generally inefficient.

This paper presents various scenarios regarding all the interactions between different portfolio frontiers and provides analytical solutions for their intersections; moreover, it introduces the "Fixed VaR-TEV Frontier" (FVTF), a new portfolio boundary which has two important properties: on the one hand, it allows asset managers to satisfy TEV and VaR restrictions at the same time and, on the other hand, it contains portfolios more efficient than those belonging to the CMTF. However, when TEV and VaR restrictions are too stringent, the FVTF does not exist. Conversely, when this frontier operates, an interesting trade-off between relative and absolute risk arises, consistent with Roll (1992). In other words, for any given expected return within the FVTF, managers can choose to reduce the relative risk (TEV) by augmenting the absolute risk (overall portfolio variance) or increase the relative risk by decreasing the absolute risk. This can also create a principal-agent problem between fund investors and asset managers.

To conclude, generalising the results to non normally distributed expected returns and disallowing short sales would surely represent the natural extensions of the analysis carried out in this paper.

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Appendix

Proof of Theorem 1

This theorem can be demonstrated from two viewpoints; from the geometric perspective, the straight line passes through portfolios $J_1(\sigma_1, \mu_1)$ and $J_2(\sigma_2, \mu_2)$, with $\sigma_1 > \sigma_2$ and $\mu_1 > \mu_2$, both belonging to the MTF. Given these assumptions, the secant line passing through segment $\overline{J_1 J_2}$ has greater slope than the asymptotic slope of the frontier. The proof consists of a comparison between the equation of the asymptotic slope of the MTF and the slope of the line passing through portfolios J_1 and J_2. Using equation (17) one can obtain

$$\frac{d}{2\Delta_1}(\sigma_1 + \sigma_2) > \sqrt{d}$$

then

$$\mu_B < \mu_C + \sqrt{d}\left(\frac{\sigma_1 + \sigma_2}{2}\right), \tag{A-1}$$

where $\mu_C + \sqrt{d}\bar{\sigma}^2$ represents the value of the asymptote of MVF calculated in $\bar{\sigma}^2 = (\sigma_1 + \sigma_2)/2$; for the convexity of the hyperbola MTF $\bar{\sigma}^2 > \sigma_B^2$, so

$$\mu_B < \mu_C + \sqrt{d}\sigma_B^2 < \mu_C + \sqrt{d}\bar{\sigma}^2.$$

This demonstrates equation (A-1).

Intersection between the MVF and the CVF

Even if Alexander and Baptista (2008) fully discussed the relationships between the MVF and the CVF, they omitted the analytic solutions for the contact portfolios (M, M_1 and M_2 in Figure 3). This section proves the existence of such analytical solutions. The condition under which the frontiers intersect in (σ_P^2, μ_P) space is the equality between equations (1) and (8)

$$\left(\frac{\mu_P + V_0}{z_\theta}\right)^2 = \sigma_C^2 + \frac{1}{d}(\mu_P - \mu_C)^2. \tag{A-2}$$

The resolvent obtained after some algebra is the following quadratic expression

$$(z_\theta^2 - d)\mu_P^2 - 2(z_\theta^2\mu_C + dV_0)\mu_P + cz_\theta^2\sigma_C^2 - dV_0^2 = 0, \tag{A-3}$$

which returns the solutions (for portfolios M_1 and M_2)

$$\mu_P = \frac{(z_\theta^2\mu_C + dV_0) \pm \sqrt{D_0}}{(z_\theta^2 - d)} \tag{A-4}$$

when the discriminant

$$\begin{aligned} D_0 &= (z_\theta^2\mu_C + dV_0)^2 - (z_\theta^2 - d)(cz_\theta^2\sigma_C^2 - dV_0^2) \\ &= V_0^2 + 2\mu_C V_0 + \sigma_C^2(c - z_\theta^2) \end{aligned} \tag{A-5}$$

is strictly positive; this occurs when

$$V_0 \geq -\mu_C + \sqrt{\sigma_C^2(z_\theta^2 - d)}, \tag{A-6}$$

which represents the solution lying in the efficient set[11] under the necessary condition $z_\theta^2 - d > 0$. However, when the confidence level is low ($z_\theta^2 - d < 0$), it can be shown that $D > 0$, hence a VaR at which the CVF crosses the MVF must surely exist.

The tangency condition $D_0 = 0$ demonstrates equation (13) related to tangency portfolio M. Moreover, substituting $D_0 = 0$ in equation (A-4), the result in equation (14) is immediately obtained.

When the tangency between the MVF and the CVF holds, some mathematical aspects have to be taken into consideration:

[11] All the algebra is omitted for brevity. It is clear that if equation (A-5) admits two real solutions, only the relationship (A-6) has to be considered; in practice, the other condition $V_0 \leq -\mu_C + \sqrt{\sigma_C^2(z_\theta^2 - d)}$ corresponds to a VaR constraint that is too stringent (see, for instance, Figure 3).

- if $z_\theta^2 < d$ (low confidence level), the radical is negative hence the solution (13) is not real and the tangency between the frontiers is not possible, as shown in Figure 5;
- if $z_\theta^2 = d$ (low confidence level), the solution is $V_0 = -\mu_C$ which consists of the maximum small bound (see Figure 3): in this case, the straight line CVF is the asymptote of the MVF, hence the tangency cannot exist for finite values of σ_P or μ_P. This situation corresponds to Proposition 1.(ii) in Alexander and Baptista (2008);
- if $z_\theta^2 > d$ (high confidence level), the tangency condition always holds when VaR=V_M in equation (13).

Intersection between the MTF and the CVF

The process of finding the intersection between the MTF and the CVF is a similar to that of the previous case: the frontiers show common portfolios in (σ_P^2, μ_P) space when

$$\left(\frac{\mu_P + V_0}{z_\theta}\right)^2 = \sigma_B + \frac{1}{d}(\mu_P - \mu_B)^2 + 2\frac{\Delta_1}{d}(\mu_P - \mu_B). \qquad \text{(A-7)}$$

The resolvent is the following second degree equation

$$(z_\theta^2 - d)\mu_P^2 - 2(z_\theta^2\mu_C + dV_0)\mu_P + z_\theta^2(d\sigma_B^2 - \mu_B^2 + 2\mu_C\mu_B) - dV_0^2 = 0, \qquad \text{(A-8)}$$

and the solutions (for portfolios R_1 and R_2) are

$$\mu_P = \frac{(z_\theta^2\mu_C + dV_0) \pm \sqrt{D_1}}{(z_\theta^2 - d)}, \qquad \text{(A-9)}$$

where the discriminant is

$$\begin{aligned} D_1 &= (z_\theta^2\mu_C + dV_0)^2 - (z_\theta^2 - d)[z_\theta^2(d\sigma_B^2 - \mu_B^2 + 2\mu_B\mu_C) - dV_0] \\ &= V_0^2 + 2\mu_C V_0 - z_\theta^2\left(\sigma_B^2 - \frac{\Delta_1^2}{d}\right) + d\sigma_B^2 - \Delta_1^2 + \mu_C^2. \qquad \text{(A-10)} \end{aligned}$$

Given that the objective of the MTF is to minimise TEV, in equation (A-10) a benchmark portfolio is considered. The tangency condition $D_1 = 0$ is satisfied when the value of the constrained VaR is

$$V_R = -\mu_C + \sqrt{\left(\sigma_B^2 - \frac{\Delta_1^2}{d}\right)(z_\theta^2 - d)}. \qquad \text{(A-11)}$$

As in the previous case, the analytical solution for this constraint depends on the data and the confidence level θ. The relationship $V_R > V_M$ indicates that this constraint is less stringent. In short:

$$V_R > V_M \;\Rightarrow\; z_\theta^2\left(\sigma_B^2 - \frac{\Delta_1^2}{d}\right) > \sigma_C^2(z_\theta^2 - d) \;\Rightarrow\; z_\theta^2 > -d\frac{\sigma_C^2}{\delta_B}.$$

This is true because $d > 0$, $\sigma_C^2 > 0$ and $\delta_B > 0$ by construction. Furthermore, the analysis conducted for $(z_\theta^2 - d) \lesseqgtr 0$, for the intersection MVF-CVF, is substantially confirmed. Finally, substituting $V_0 = V_R$ in equation (A-8), the resulting portfolio is R, as already defined in equation (16).

Intersection between the CTF and the CVF: the system

Starting from equations (4) and (8), the intersections between the CTF and the CVF correspond to the solutions of the system

$$\begin{cases} x = \left(\dfrac{\mu_P + V_0}{z_\theta}\right)^2 - A_0 \\ \\ dx^2 + \phi_1(\mu_P - \mu_B)^2 + \phi_2 x(\mu_P - \mu_B) + \phi_3 = 0 \end{cases} \tag{A-12}$$

where $A_0 = \sigma_B^2 + T_0$, $x = \sigma_C^2 - A_0$, $\phi_1 = 4\Delta_2$, $\phi_2 = -4\Delta_1$ and $\phi_3 = 4d\delta_B T_0$. The resolvent is provided by the quartic function

$$d\left[\left(\frac{\mu_P + V_0}{z_\theta}\right)^2 - A_0\right]^2 + \phi_1(\mu_P - \mu_B)^2 + \phi_2\left[\left(\frac{\mu_P + V_0}{z_\theta}\right)^2 - A_0\right](\mu_P - \mu_B) + \phi_3 = 0. \tag{A-13}$$

Equation (9) is obtained after some algebra, thus:

$$\begin{cases} c_1 = \dfrac{d}{z_\theta^4} \\ \\ c_2 = 4\dfrac{d}{z_\theta^4}V_0 + \dfrac{\phi_2}{z_\theta^2} \\ \\ c_3 = 6\dfrac{d}{z_\theta^4} - 2\dfrac{d}{z_\theta^2}A_0 + \phi_1 + 2\dfrac{\phi_2}{z_\theta^2}V_0 - \dfrac{\phi_2}{z_\theta^2}\mu_B \\ \\ c_4 = 4\dfrac{d}{z_\theta^4}V_0^3 - 4\dfrac{d}{z_\theta^2}A_0 V_0 - 2\phi_1\mu_B^2 - \phi_2 A_0 + \dfrac{\phi_2}{z_\theta^2}V_0(V_0 - 2\mu_B) \\ \\ c_4 = \dfrac{d}{z_\theta^4}V_0^4 - 2\dfrac{d}{z_\theta^2}A_0(V_0^2 + dA_0) + \phi_1\mu_B^2 - \phi_2\mu_B\left(\dfrac{V_0^2}{z_\theta^2} - A_0\right) + \phi_3 \end{cases}$$

System (A-12) should return two distinct real solutions for the expected return μ_P when the parabola CVF crosses the ellipse in (σ_P^2, μ_P) space, a double root when the curves are tangent and no solutions when the frontiers do not have any portfolios in common. This implies that, when the intersections occur, the polynomial of the fourth degree (9) always has two non real complex conjugate roots; this result is very difficult to handle, so the contacts between the frontiers are determined via the numerical optimisation method introduced in section 3.

Proof of equation (10)

Setting $x = \sigma_P^2 - \sigma_B^2 - T_0$, the function (4) becomes the following second order equation

$$dx^2 - 4\Delta_1(\mu_P - \mu_B)x + 4\Delta_2(\mu_P - \mu_B)^2 - 4dT_0\delta_B = 0. \qquad \text{(A-14)}$$

Solving x, after some algebra, one can obtain

$$x = \frac{2}{d}\left\{\Delta_1(\mu_P - \mu_B) \pm \sqrt{d\delta_B[dT_0 - (\mu_P - \mu_B)^2]}\right\}.$$

Given that the CVF intersect the CTF on the left, the smaller solution of equation (A-14) has to be considered, therefore the function

$$\sigma_P^2 = \sigma_B^2 + T_0 + \frac{2}{d}\left\{\Delta_1(\mu_P - \mu_B) - \sqrt{d\delta_B[dT_0 - (\mu_P - \mu_B)^2]}\right\} \qquad \text{(A-15)}$$

defines $S_T(\mu_P)$ in equation (10). On the other hand, equation $S_V(\mu_P)$ derives from the definition of the straight line (8).

First order condition of equation (10)

When $\mu_2 < \mu_P < \mu_1$, the first order condition is

$$\begin{aligned}\frac{\partial S(\mu_P)}{\partial \mu_P} &= 2[S_T(\mu_P) - S_V(\mu_P)]\left[\frac{\partial S_T(\mu_P)}{\partial \mu_P} - \frac{\partial S_V(\mu_P)}{\partial \mu_P}\right] \\ &= 4[S_T(\mu_P) - S_V(\mu_P)]\left[\frac{\Delta_1}{d} + \frac{\sqrt{\delta_B/d}(\mu_P - \mu_B)}{\sqrt{dT_0 - (\mu_P - \mu_B)^2}} - \frac{\mu_P + V_0}{z_\theta^2}\right].\end{aligned}$$

If $V_0 \geq V_K$, the first polynomial is zero when $\mu_P = \mu_{K_1}$ or $\mu_P = \mu_{K_2}$, where K_1 and K_2 are the portfolios at which the CTF and the CVF intersect, while the second polynomial admits a unique solution at $\mu_P = \mu_P^*$, where $\mu_{K_1} \leq \mu_P^* \leq \mu_{K_2}$ (the strict equality $\mu_{K_1} = \mu_P^* = \mu_{K_2}$ holds when $V_0 = V_K$). According to this results, the function $S(\mu_P)$ shows two minima and a local maximum placed between them; heuristically, this property is demonstrated by Figure 2 (c).

When $V_0 < V_K$, the CTF and the CVF do not show common portfolios, so the first order condition is only guaranteed by the sole solution $\mu_P = \mu_P^*$, which corresponds to the expected return that minimise the distance between frontiers. Under this assumption, $S(\mu_P)$ is strictly convex.

Proof of equation (18)

Given $M \equiv (\sigma_C^2 + d\sigma_C^2/(z_\theta^2 - d), \mu_C + d\sigma_C/\sqrt{z_\theta^2 - d})$ and $H \equiv (\sigma_C^2 + \Delta_1^2/d, \mu_B)$, if $M \equiv H$, it follows that

$$\mu_C + \frac{d\sigma_C}{\sqrt{z_\theta^2 - d}} = \mu_B \quad \Rightarrow \quad \sqrt{z_\theta^2 - d} = \frac{d\sigma_C}{\Delta_1}.$$

Given that $d > 0$, $\sigma_C > 0$, $\Delta_1 > 0$ and $z_\theta^2 - d > 0$, by definition, the solution is

$$z_\theta^H = \sqrt{d + \frac{d^2 \sigma_C^2}{\Delta_1^2}}$$

Moreover, the relationship $z_\theta^H > \sqrt{d}$ is straighforward:

$$z_\theta^H = \sqrt{d}\sqrt{\frac{\Delta_1^2 - d\sigma_C^2}{\Delta_1^2}}$$

Proof of equation (19)

From equations (17) and (18) it follows that

$$\frac{d}{2\Delta_1}(\sigma_1 + \sigma_2) > \sqrt{d + \frac{d^2 \sigma_C^2}{\Delta_1^2}}$$

therefore

$$\frac{\sigma_1 + \sigma_2}{2} > \sqrt{\sigma_C^2 + \frac{\Delta_1^2}{d}} = \sigma_H.$$

Given that portfolio H lies on the Mean-Variance Frontier, it surely has a lower risk than the average of risks in portfolios J_1 and J_2, and this completes the proof.

Package

All the routines for the analysis carried out in this paper, together with the entire data set used, are freely downloadable from

`http://utenti.dea.univpm.it/palomba/TEV-VaR.html`

A short user guide is also available.

Table A-1: Asset classes

Asset class	Expected Return (%)	Std Dev. (%)	Correlations Auto	Bank	Chem.	Cons.	Ener.	Ind.	Ins.	Tel.	Util.	Oth.
Automobiles	4.068	15.620	1.000	-	-	-	-	-	-	-	-	-
Banks	0.953	16.532	0.661	1.000	-	-	-	-	-	-	-	-
Chemicals	2.993	9.858	0.606	0.708	1.000	-	-	-	-	-	-	-
Constructions	3.353	12.449	0.564	0.725	0.729	1.000	-	-	-	-	-	-
Energy	1.412	8.822	0.592	0.715	0.768	0.727	1.000	-	-	-	-	-
Industrial	3.674	9.682	0.696	0.784	0.805	0.749	0.750	1.000	-	-	-	-
Insurance	1.119	16.970	0.698	0.845	0.819	0.768	0.748	0.777	1.000	-	-	-
Telecommunications	0.597	10.199	0.440	0.661	0.620	0.722	0.727	0.650	0.672	1.000	-	-
Utilities	2.788	10.873	0.512	0.698	0.731	0.749	0.843	0.707	0.729	0.761	1.000	-
Other	0.630	13.171	0.658	0.855	0.661	0.629	0.558	0.690	0.744	0.539	0.545	1.000
DJ Eurostoxx 50	0.985	10.004	0.726	0.887	0.893	0.838	0.882	0.887	0.918	0.788	0.864	0.779
Standard & Poor 500	1.484	8.510	0.687	0.829	0.791	0.774	0.799	0.866	0.826	0.761	0.775	0.686

The expected return of asset class j is calculated via the formula $R_j = \frac{1}{n_j} \sum_{i=1}^{nj} R_i$, where n_j is the number of stocks included in class j

ASSET CLASSES

Automobiles: Daimler, Renault, Volkswagen

Banks: Banco Santander, BBV Argentaria, BNP Paribas, Crédit Agricole, Deutsche Bank, Fortis, Intesa Sanpaolo, Société Générale, Unicredit

Chemicals: Air Liquide, Basf, Bayer, Sanofi Aventis

Constructions: Saint Gobain, Vinci

Energy: Enel, ENI, Repsol, GDF Suez, Total

Industrial: Arcelor Mittal, Danone, L'Oréal, LVMH, Philips, SAP, Schneider Electric, Siemens, Unilever

Insurance: Aegon, Allianz, AXA, Generali, Ing Groep, Münchener Ruck.

Telecommunications: Alcatel, Deutsche Telekom, France Telecom, Nokia, Telecom Italia, Telefonica, Vivendi

Utilities: E.On, Iberdrola, RWE

Other: Carrefour (Retail), Deutsche Börse (Financial services)

Table A-2: Empirical results (Benchmark portfolio: DJ Eurostoxx 50 Index)

Portfolio Expected Return: 5.000
TEV constraint (T_0): 20.000
Δ_1: -0.365, Δ_2: 57.383
Tangency TEV (T_H): 57.328, Ψ: -87.520
The benchmark is not extreme ($T_0 < T_R$, $V_K < V_R$)

Intermediate Bound:
VaR constraint (V_0): 15.000
High confidence level, θ: 99%, z_θ: 2.326, Threshold ($\sqrt{d}$): 1.531
$\hat{V} = V_2$: 31.579
Intersections in (σ_P, μ_P) space: $M_1 \equiv (18.231, 27.411)$ and $M_2 \equiv (6.570, 0.285)$
Intersections in (σ_P, μ_P) space: $K_1 \equiv (9.585, 7.299)$ and $K_2 \equiv (7.256, 1.880)$
Efficiency loss: δ_{K_1}: 34.098, δ_{K_2}: 9.846

Portfolios:	P	T	J	AB	B	C	Q	J_1	J_2	H	M	K	R
Exp. Return	5.000	5.000	5.000	5.000	0.985	1.350	75.494	7.833	-5.863	0.985	10.097	5.012	14.739
Variance	48.369	105.700	63.961	73.911	100.070	42.687	2387.3	117.940	122.191	42.743	75.317	64.043	176.470
Risk	6.955	10.281	7.997	8.597	10.004	6.533	48.860	10.860	11.054	6.538	8.679	8.003	13.284
Sharpe Ratio	0.719	0.486	0.625	0.582	0.098	0.207	1.545	0.721	-0.530	0.151	1.163	0.626	1.109
Alpha	4.015	4.015	4.015	4.015	-	0.365	74.509	6.848	-6.848	-	9.111	4.027	13.753
TEV	64.203	6.874	20.000	166.280	-	57.385	2425.1	20.000	20.000	57.328	92.736	20.000	80.674
Information Ratio	0.063	0.584	0.201	0.024	-	0.0064	0.0307	0.342	-0.342	-	0.098	0.201	0.170
Efficiency Loss	-	57.328	15.592	25.542	57.328	-	-	57.328	57.328	-	-	15.636	57.328
VaR	11.179	18.917	13.605	15.000	22.287	13.849	38.172	17.431	31.579	14.224	10.093	13.605	16.165

Large Bound:

VaR constraint (V_0): 15.00, with $V_0 > \mu_C$
Low confidence level, θ: 0.937, Threshold $z_\theta = \sqrt{d}$: 1.531
V_1: 8.793, V_2: 22.786
$\mu_P > \mu_R \Rightarrow AB = T$

Portfolios:	AB	M	K_2	K	R
Exp. Return	5.000	-3.764	-2.723	6.193	0.346
Variance	105.700	53.842	64.289	74.477	100.44
Risk	10.281	7.338	8.018	8.630	10.022
Sharpe Ratio	0.486	-0.513	-0.340	0.718	0.034
Alpha	4.015	-4.750	-3.708	5.208	0.639
TEV	6.874	66.949	20.000	20.000	0.174
Information Ratio	0.584	-0.071	-0.185	0.260	0.367
Efficiency Loss	57.328	-	14.529	21.786	57.328
VaR	10.742	15.000	15.000	7.021	15.000

Table A-3: Empirical results (Benchmark portfolio: Standard & Poor 500 Composite Index)

Portfolio Expected Return: 5.000
TEV constraint (T_0): 20.000
Δ_1: 0.147, Δ_2: 37.176
Tangency TEV (T_H): 37.130, Ψ: -8.141
The benchmark is not extreme ($T_0 > T_R$, $V_K < V_R$)

Intermediate Bound:
VaR constraint (V_0): 15.000
High confidence level, θ: 99%, z_θ: 2.326, Threshold ($\sqrt{d}$): 0.689, $\hat{V} = V_2$
Intersections in (σ_P, μ_P) space: $M_1 \equiv (9.044, 6.040)$ and $M_2 \equiv (5.937, 1.330)$
Intersections in (σ_P, μ_P) space: $K_1 \equiv (6.353, -0.221)$ and $K_2 \equiv (6.514, 0.154)$
Efficiency loss - δ_{K_1}: 15.657, δ_{K_2}: 4.239
z_θ^H: 19.204
$\theta^* \approx 1$, $z_\theta^* \approx +\infty$, V^*: 297.248

Portfolios:	P	T	J	AB	B	C	Q	J_1	J_2	H	M	K	R
Exp. Return	5.000	5.000	-	5.000	1.484	1.337	13.870	4.567	-1.599	1.483	2.606	2.299	3.156
Variance	63.485	100.620	-	73.911	72.423	35.247	365.760	94.330	90.515	35.293	38.641	40.365	79.345
Risk	7.968	10.031	-	8.597	8.510	5.937	19.125	9.712	9.514	5.941	6.216	6.353	8.908
Sharpe Ratio	0.628	0.498	-	0.582	0.174	0.225	0.725	0.470	-0.168	0.250	0.419	0.362	0.354
Alpha	3.516	3.516	-	3.516	-	-0.147	12.386	3.083	-3.083	-	1.123	0.815	1.673
TEV	63.147	26.017	-	112.920	-	37.175	359.940	20.000	20.000	37.130	39.783	20.000	5.888
Information Ratio	0.056	0.135	-	0.031	-	-0.004	0.034	0.154	-0.154	-	0.028	0.041	0.284
Efficiency Loss	-	37.130	-	10.426	37.130	-	-	37.130	37.130	-	-	3.170	37.130
VaR	13.536	18.335	-	15.000	18.314	12.475	30.621	18.028	23.732	12.337	11.854	12.481	17.566

Large Bound:

VaR constraint (V_0): 5.00, with $V_0 > \mu_C$
Low confidence level, θ: 0.755, Threshold $z_\theta = \sqrt{d}$: 0.689
V_1: 2.129, V_2: 8.1583
$\mu_P > \mu_R \Rightarrow AB = T$

Portfolios:	AB	M	K_2	K	R
Exp. Return	5.000	-0.510	-0.288	3.875	0.883
Variance	100.620	42.421	46.720	59.503	72.811
Risk	10.031	6.513	6.835	7.714	8.533
Sharpe Ratio	0.498	-0.078	-0.042	0.502	0.103
Alpha	3.516	-1.993	-1.771	2.391	-0.601
TEV	26.017	45.492	20.000	20.000	0.760
Information Ratio	0.135	-0.043	-0.089	0.120	-0.791
Efficiency Loss	37.130	-	5.920	10.698	37.130
VaR	10.742	5.000	5.000	1.443	5.000

Summary

Table A-4: Hierarchy of VaR constraints ($T_0 \leq T_H$)

	Low confidence level		
	small bound	minimum bound	strong bound
$0 < z_\theta < \sqrt{d}$	-	-	$V_M = V_R < V_K < V_1 < V_2$
$z_\theta = \sqrt{d}$ (*)	no solutions ($V_M < -\mu_C$)	no solutions ($V_M = -\mu_C$)	$-\mu_C < V_M < V_K < V_1 < V_2$
	medium bound	intermediate bound	maximum bound
$0 < z_\theta < \sqrt{d}$	$V_M = V_R = V_K < V_1 < V_2$	$V_K < V_M = V_R < V_1 < V_2$	$V_K < V_M = V_R = V_1 < V_2$
$z_\theta = \sqrt{d}$ (*)	$-\mu_C < V_M = V_K < V_1 < V_2$	$-\mu_C < V_K < V_M < V_1 < V_2$	$-\mu_C < V_K < V_M = V_1 < V_2$
	large bound	larger bound	no bound
$0 < z_\theta < \sqrt{d}$	$V_k < V_1 < V_M = V_R < V_2$	$V_K < V_1 < V_M = V_R = V_2$	$V_K < V_1 < V_2 < V_M = V_R$
$z_\theta = \sqrt{d}$ (*)	$-\mu_C < V_K < V_1 < V_M < V_2$	$-\mu_C < V_K < V_1 < V_M = V_2$	$-\mu_C < V_K < V_1 < V_2 < V_M$
	High confidence level		
$\Delta_1 > 0$	all bounds	$\Delta_1 \leq 0$	all bounds
$\sqrt{d} < z_\theta < z_\theta^H$	$V_M < V_K < V_H < V_R < V_1 < V_2$	$z_\theta > \sqrt{d}$	$V_M < V_K < V_H < V_R < V_1 < V_2$
$z_\theta = z_\theta^H$	$V_M \leq V_K \leq V_H < V_R < V_1 < V_2$		
$z_\theta^H < z_\theta < z_\theta^*$	$V_M < V_K < V_H < V_R < V_1 < V_2$		
$z_\theta = z_\theta^*$	$V_M < V_K < V_H < V_R < V_1 = V_2$	extreme benchmarks	
$z_\theta > z_\theta^*$	$V_M < V_K < V_H < V_R < V_2 < V_1$	$z_\theta > \sqrt{d}$, $T_0 < T_R$	$V_M < V_H < V_R < \hat{V}$, where $V_K \lesseqgtr V_R$

(*) When $V_M > \gamma_B = \mu_C - \sqrt{d\delta_B}$, where γ_B is the intercept of the MTF asymptote, it follows that $V_M = V_R$.