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MINIMUM TRACKING ERROR VOLATILITY

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Abstract

Investors assign part of their funds to asset managers that are given the task of beating a benchmark. The risk management department usually imposes a maximum value of the tracking error volatility (TEV) in order to keep the risk of the portfolio near to that of the selected benchmark. However, risk management does not establish a rule on TEV which enables us to understand whether the asset manager is really active or not and, in practice, asset managers sometimes follow passively the corresponding index. Moreover, the benchmark is sometimes difficult to be beaten when the risk managers only check that portfolio managers do not exceed a fixed level of relative risk.

I derive analytical methods that could be used to understand whether the strategy used by the portfolio manager is active that allows him/her to have an excess return above the benchmark large enough to cover the commission paid by investors and, concurrently, that allows him/her to restrict the portfolio's variance to be not more than the benchmark's variance in order to avoid an excess return merely due to a higher risk level (using variance as risk indicator). These equations are a necessary (but not sufficient) condition to beat the benchmark's return, without increasing the overall variance of the portfolio. This is also a generalization of the model of Jorion (2003) with the use of commissions.

I apply these equations to an Italian liquidity fund and I find that the fees are too high and the TEV is low. In fact, all the funds in the liquidity category show similar problems that often render the portfolio unable to cover the fees without increasing the variance.

Keywords: Active Management, Tracking Error, Benchmarking, Commissions, Portfolio Choice, Risk Management.

JEL classification codes: G11, G10, G23, C61.

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1 Introduction

Investors sometimes assign part of their funds to an asset manager, who is given the task of beating a benchmark. In fact, asset managers, that receive a fee from investors to beat the benchmark, often follow passively their index. Moreover, risk managers that only check that portfolio managers do not exceed a fixed level of relative risk, often make the benchmark difficult to be beaten. The comparatively poor performance of asset managers is one of the main reasons of the decreasing growth of the Investment Fund market. The world asset management market faced a big reduction of its growth in 2008 and in the first half of 2009, as it can be seen in figure 1. Moreover analyzing only the "long-term" part (all funds excluding Money market ones) in 2008 the net sales[1] were largely negative (over 400 Euro billions).
Figure 2 shows that the health of Europe market is even worse, indeed it records a large fall of net sales since the second half of 2007, with only a small rescue in 2009, after the crash in october 2008 (-130 Euro billions in only one month due to the financial crisis also tied to the Lehman's Brother bankruptcy in September 2008).
Focusing on single nations it is evident that some very critical situations exist. For example in Italy investors ransomed a huge amount of money from open funds over the last few years: in 2008 net sales were negative at over 143 billion euro, however the decrease of interest for open funds began since 2002 and increased since 2006, as it can be seen in fig.3.
One of the problem of the asset management market is the presence of Exchange Traded Funds (ETFs), that are a strong competitor of funds due to the fact that they posses some advantages that are fundamental especially during critical periods: low fees, better liquidity and better disclosures. The global ETF market grew very much in these years, also during the crisis; in 2009 the global asset under management exceeded 1000 billions dollars, with a +45% that is much more than the +27% of the Msci World index[2].
Now, against ETFs, all the funds that follow a passive strategy have no longer reason to exist, and also the need to outperform the benchmark is strongly increased for active manager[3]. For this reason I want to understand if a manager is truly active (he/she has the possibility to beat the benchmark without increasing the overall portfolio variance) and, to do this, I can use a measure already employed in the asset and risk management departments: the tracking error volatility (**TEV**), which measures the volatility of the deviations of an active portfolio's

[1]New sales of funds plus reinvested dividends less redemptions.

[2]Source for the global ETF market data are the BlackRock Etf Landscape Industry Review and the EDHEC European Etf Survey.

[3]For Example, Nizam Hamid, the sale director of iShares Europe, calculates that in the 2001-2008 period, ETFs overperformed their benchmark by 89 basis points, with a volatility of 66 basis points under their index. Instead, active funds had an average underperformance of 297 basis points less than their benchmarks, with a volatility higher than the benchmarks themselves (source: Bluerating). Also French (2008) finds that an U.S. investor would increase his average annual return by 67 basis points over the 1980 to 2006 period if he/she switched to a passive market portfolio.

Figure 1: *Net Sales of Funds worldwide (Euro billions). Source: EFAMA.*

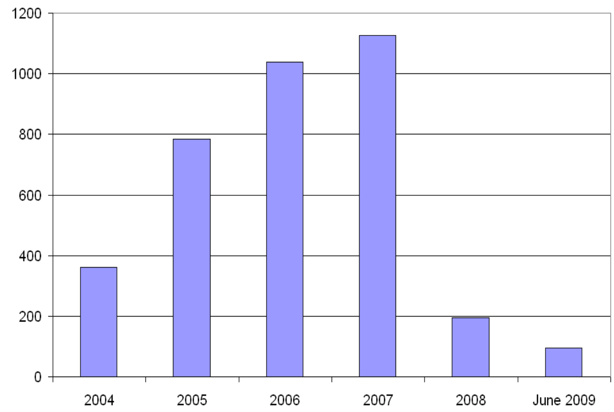


Figure 2: *Net sales of European UCITS (Euro billions). Source: EFAMA.*

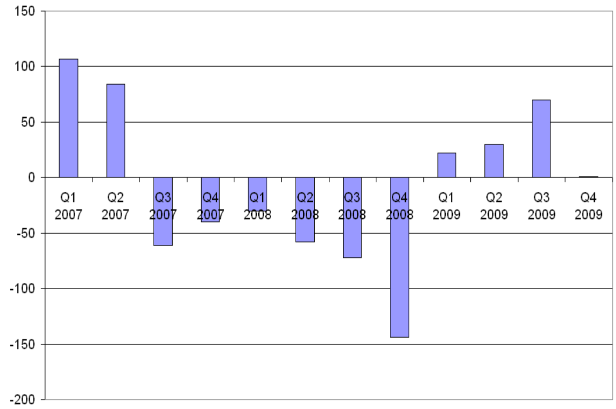
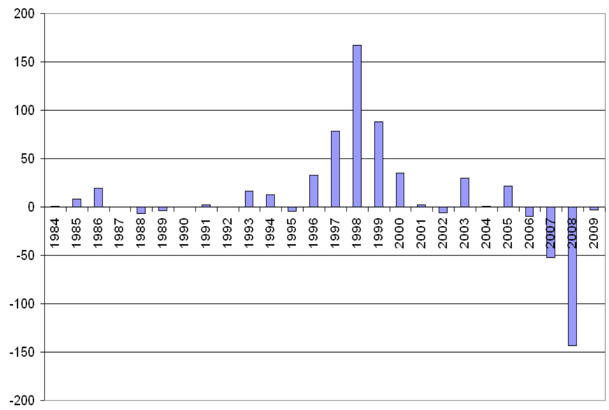


Figure 3: *Net Sales of Italian Open Funds (Euro billions). Source: Assogestioni.*



returns from the benchmark's returns.
The paper begins with a section that introduces the TEV measure; in section 3 the literature review is presented; the analytical derivation of the minimum TEV value is in section 4; section 5 applies the obtained formulas and section 6 concludes.

2 Tracking Error Volatility

I analyse one of the most popular techniques to impose a restriction on the activity of an asset manager: the use of a limit on the Tracking Error Volatility (**TEV**). The TEV measures the volatility of the deviations of an active portfolio's returns from the benchmark's returns. I define the following notation, as done by Jorion (2003)[4]:

- $\mathbf{q}$: vector of benchmark weights;
- $\mathbf{x}$: vector of deviations from benchmark;
- $\mathbf{q}_p = \mathbf{q} + \mathbf{x}$: vector of portfolio weights;
- $\mathbf{e}$: vector of expected returns;
- $\mathbf{V}$: covariance matrix of asset returns.

So I can write the TEV as:

$$T = \mathbf{x'Vx} \tag{1}$$

This measure can be both backward or forward-looking[5]: it depends on the use of x as past or current deviations and V as occurred or guessed covariance matrix. As in Jorion (2003) I will use the TEV in the forward-looking meaning, but it would be the same in the backward-looking sense (and in the empirical application the backward-looking sense will be used).

The tracking error has different uses, depending on the strategy of the portfolio:

- a passive strategy seeks to reproduce a benchmark portfolio return, by minimizing the tracking error;
- an active strategy seeks to outperform an index, while staying within certain relative risk boundaries defined by the TEV.

[4]The notation is the same as Jorion (2003), but not to confuse the reader, I will use, differently from Jorion, bold character for vectors and $\mathbf{e}$ instead of E for the vector of expected returns: in this way the reader can distinguish a scalar from a vector (in bold) and from a matrix (in capital and bold). However, I point out that only the TEV, following Jorion, in the equations is shortened in T, capital letter, even if it is a scalar value.

[5]The forward-looking TEV can be linked with the concept of Value-at-Risk (VaR) as shown by Jorion (2003). Recently many papers focused on VaR constraint in portfolio selection, as Gordon and Baptista (2008) or Campbell *et al.* (2001).

Now passive strategies, as already said, are developed directly by computers that manage the Exchanged Traded Funds, so asset managers have to implement active strategies. For this reason I focus only on the active strategy.
In active portfolios, it is well-known that TEV is used by risk managers to give to asset managers an upward limit of relative risk, which helps to preserve investors' funds from dangerous bets. In this paper I will upset the aim of this measure, indeed it could be also used as a downward limit to understand if the portfolio manager truly works to beat the benchmark, or if he/she follows the index passively.

The tracking error volatility is not the only way to analyse the relative risks that a manager assumes and examples of alternative measures are in Rudolf *et al.* (1999)[6]. However, I will focus on TEV, because it is in practice the most used measure of relative risk. An extension to other relative risk measures could be done in future studies.

3 Literature review

The literature on portfolio management is vast, with research papers starting from Markowitz (1952). A lot of authors analyze the implications of constraints on the allocations, for example Jagannathan and Ma (2003) explain why constraints are useful. Other authors, as Boyle and Tian (2007), do not study the problem of maximizing the expected utility of terminal wealth, but the problem of outperforming a benchmark using constraints. The tracking error volatility is the constraint on relative risk more used in practice and the framework that defines the investment goal in terms of the excess return on benchmark and defines risk in term of tracking error starts with papers like Franks (1992). In the field of benchmarking, a lot of studies debate various topics. One topic is how to optimize costs in passive management choosing the best number of asset to use, as Jansen and van Dijk (2002), or choosing when to rebalance, as Gaivoronski *et al.* (2005). Another topic is how an active manager can beat

[6]In Rudolf *et al.* (1999), the authors compare the TEV with four other measures:

1. the mean absolute deviations (MAD) of portfolio's returns compared to the benchmark's one;
2. the mean absolute downside deviations (MADD), that is the MAD restricted to negative deviations between portfolio and benchmark returns;
3. the MinMax, that makes the portfolio weights such that the maximum deviation between portfolio and benchmark returns is minimized;
4. the Downside MinMax (DMinMax), that is the MinMax restricted to negative return deviations.

Rudolf *et al.* (1999) think that a linear tracking error measure is better because fund managers have linear performance fees, so this model improves the description of their risk attitude. Moreover, they show that a linear tracking error optimization is equivalent to expected utility maximization. However, they try an empirical application and their results do not show that MAD or MADD models give weights (and so returns and standard deviations) very different from those obtained with an algorithm that minimizes the TEV.

the benchmark using a particular division of labour, as suggested in papers like Lee (2000), or using particular strategies, as in Browne (1999). This topic is directly related with the evaluation of the asset manager and also in this field a lot of authors, like Clarke *et al.* (2002), try to discern the ability of the asset manager to forecast returns from a portfolio performance that is affected by constraints or try to understand the source of the "alpha"[7] produced by the asset manager, like Lo (2007), or try to understand how active is the asset manager, as Cremers and Petajisto (2009). Another literature layer is about management fees, as Kritzman (1987) or Elton *et al.* (2003). Authors, like Kuenzi (2004), focus on tactical ranges on portfolio weights to assure the compliance of tracking error limit. Recently, some authors, like Barro and Canestrelli (2008), are analyzing how to optimize portfolios that have structured pay-offs, as with a minimum guarantee constraint.

Our interest is focused on the tracking error literature generated by Jorion (2003), that brings back the excess of return-TEV problem in the mean-variance space. A lot of authors use this methodology: some authors, like El-Hassan and Kofman (2003), add further constraints such as no short-selling, while other authors, as Palomba (2008), empirically analyse methods of optimizing portfolios. Now Jorion's paper is better reviewed.

3.1 A review of Jorion (2003)

Jorion notes that the excess return-TEV problem induces the manager to optimize in an excess return space only, ignoring the investor's overall portfolio risk[8]. Roll (1992) already noted that excess return optimization leads to portfolio with a sistematically higher risk than the benchmark and also other authors, as Scowcroft and Sefton (2001), questioned the utility of a constraint on TEV. It could be more important to compare the overall volatility of the portfolio with the volatility of the benchmark (assuming variance as a measure of risk) instead of using the TEV to value the risks undertaken by the asset manager[9]. However, the asset management industry maintains emphasis on tracking error risk control, therefore Jorion (2003) tries to correct the problem of portfolios with higher variance than the benchmark, imposing an additional constraint (beyond the maximum TEV one) that forces the portfolio volatility to be equal to that of the index: $\sigma_P^2 = \sigma_B^2$.

Before imposing this further restriction, Jorion analyses the form of the TEV-constrained frontier in the mean-variance space. To summarize Jorion's results, I need to use the notation

[7]Running an OLS regression of portfolio's return on benchmark's return, an high intercept "alpha" represents the ability of the asset manager in choosing single stocks or bonds that have a return higher than the index or in having a goog market timing.

[8]Ineichen (2004) focus also on the poor returns of the relative performance paradigm.

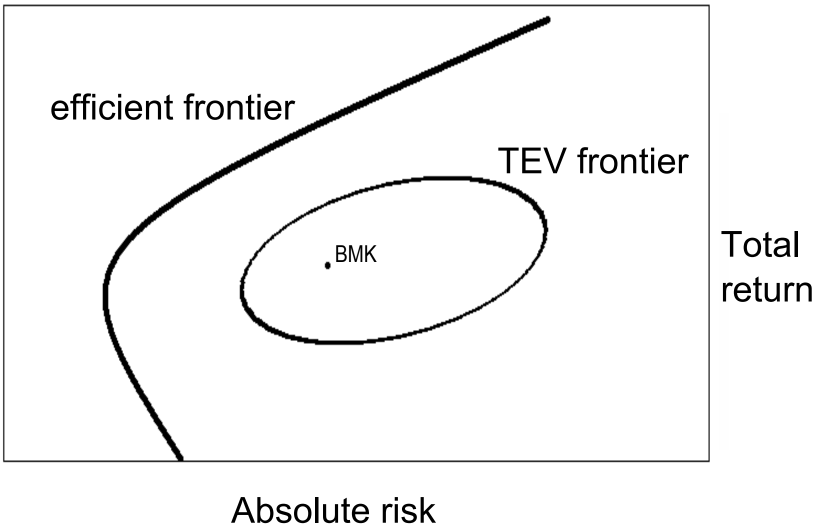
[9]Moreover Admati and Pfleiderer (1997) argue that the rationale for benchmark-adjusted compensation scheme is inconsistent with optimal risk-sharing and does not help in solving potential contracting problems with the manager. Many studies were done on the agent/manager relationship and Cornell and Roll (2005) also show that delegated investing should be considered by the capital market equilibrium theories.

introduced in section 2, and also to define:

- $\mu_B = \mathbf{q'e}$: benchmark's expected return;
- $\sigma_B^2 = \mathbf{q'Vq}$: expected variance of benchmark's returns;
- $\mu_\epsilon = \mathbf{x'e}$: portfolio's expected excess return;
- $\mu_P = (\mathbf{q} + \mathbf{x})'\mathbf{e} = \mu_B + \mu_\epsilon$: portfolio's expected return;
- $\sigma_P^2 = (\mathbf{q} + \mathbf{x})'\mathbf{V}(\mathbf{q} + \mathbf{x})$: portfolio's expected variance.

Jorion finds that the constant TEV frontier is an ellipse in the mean-variance space (fig.4).

Figure 4: *Efficient frontier and constrained tracking error volatility frontier.*



Now I summarize some features of the ellipse, analysed by Jorion (2003), that I need to remember in order to face our problem. A feature is that the horizontal center of the ellipse is displaced to the right, compared to the benchmark, by the amount of tracking error variance T, indeed it is centered at μ_B and $\sigma_B^2 + T$. Another characteristic is that the main axis of the ellipse has a positive slope when $\mu_B - \mu_{MV} > 0$, a negative slope when $\mu_B - \mu_{MV} < 0$ and is horizontal when $\mu_B = \mu_{MV}$; μ_{MV} is the expected return of the minimum variance portfolio and it is equal to $-b/c$, with $b = \mathbf{e'V}^{-1}\mathbf{1}$ and $c = \mathbf{1'V}^{-1}\mathbf{1}$ ($\mathbf{1}$ is a vector of 1), following Jorion's notation. Another feature is:

$$\sigma_P^2 = \sigma_B^2 + T + 2\Delta_1\sqrt{\frac{T}{d}} \tag{2}$$

always with Jorion's notation, $d = a - b^2/c$ with $a = \mathbf{e'V}^{-1}\mathbf{e}$ and b and c as defined above, while $\Delta_1 = \mu_B - \mu_{MV}$.
Finally I report the constant TEV frontier that is given by:

$$dy^2 + 4\Delta_2 z^2 - 4\Delta_1 zy - 4T(d\Delta_2 - \Delta_1^2) = 0 \tag{3}$$

defining the deviations from the center as $y = \sigma_P^2 - \sigma_B^2 - T$ and $z = \mu_P - \mu_B$, and $\Delta_2 = \sigma_B^2 - \sigma_{MV}^2$, where the variance of the minimum variance portfolio is $\sigma_{MV}^2 = -\frac{1}{c}$. This represents an ellipse when $d\Delta_2 - \Delta_1^2 > 0$, that is the case when the benchmark is within the efficient set. After the analysis of the constant TEV frontier in mean-variance space, Jorion, as already said, applies another constraint to the optimisation problem: $\sigma_P^2 = \sigma_B^2$. Then he evaluates if it is "costly" in terms of a drop in μ compared to a drop in σ, and he finds that this ratio is very low, thanks to the shape of the ellipse that is rather flat in the upper part. Additionally he shows that this constraint is more useful if the TEV is low or if the benchmark is more inefficient.

El-Hassan and Kofman (2003) applies Jorion's methodology on a portfolio of Australian stocks, adding a short selling constraint. They observe some features with the short-selling constraint:

- the opportunity set is severely reduced;
- assets weights are smoother, which implies less rebalancing costs;
- ex-post TEV does not usually exceeds the ex-ante one, while ex-post unconstrained TEV heavily exceeds the ex-ante one, with a difference that is increasingly higher as maximum tracking error is increasingly higher.

I will now use Jorion's methodology to develop a rule to evaluate if a manager can be considered active or not.

4 Minimum TEV in presence of commissions

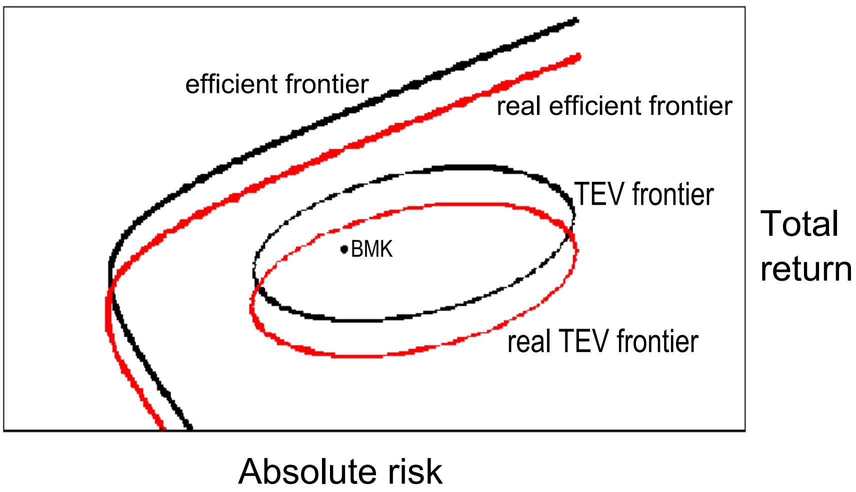
In an active strategy, the declaration of a benchmark is relevant for two purposes:

1. the definition of the risk profile of the portfolio;
2. the evaluation of the performance of the fund's manager.

Risk managers usually impose a maximum TEV to keep the risk of the portfolio near to the risk of the selected benchmark. However, in doing so, a rule on TEV is not established to understand if the asset manager is truly active or not. I want to fill this gap with a rule that

also fixes a minimun level of TEV, so the portfolio manager should have to keep the fund's TEV in a range. Therefore I calculate the minimum TEV that an asset manager has to face if he/she wants to reach the return of his/her benchmark, considering that the portfolio manager receives a fee that reduces the return of the fund, lowering all the efficient frontier (with and without TEV constraint); this fee produces a downward translation of the frontier by the amount of the commission, as shown in fig.5. For example a fund that is a perfect copy of the benchmark has a return under the return of the benchmark equal to the amount of the fee.
In this case I am considering management fees paid to the asset management company for its work, that are a common feature of all investment funds, and not subscription fees or performance fees.

Figure 5: *Efficient frontier and constrained tracking error volatility frontier with fee (red) and without fee (black).*



4.1 Mathematical proof

The notation introduced in sections 2 and 3.1 is used to set up our optimization problem. I want the minimum TEV:

$$\min \frac{1}{2}\mathbf{x'Vx} \tag{4}$$

Subject to these constraints:
$\mathbf{x'1} = 0$
$\mathbf{x'e} = com$
$\sigma_P^2 \leq \sigma_B^2$

where *com* is the amount of fees paid by the investors to the asset manager.
The first constraint is straightforward and imposes that the portfolio deviations (bets) must add up to zero.
The second constraint is the core of this study: it asks to the portfolio manager to cover the management fees with his/her active portfolio selection. It could be written as:

$$\mathbf{x'e} = \text{excess return} \tag{5}$$

This is a more general expression: the excess return could be more than the fee, for example if I want to beat the benchmark at least by a minimum amount[10], or could be less, for example it could be the difference between the commission of the portfolio and the fee of an ETF that invests on the same market. However, now the focus is in covering management fees, so I will use the notation *com*.
The third constraint imposes that the portfolio's variance should not be more than the benchmark's variance, in order to avoid an excess return of the portfolio above the index merely due to a higher risk level. This equation can be rewritten as $\mathbf{x'Vx} + 2\mathbf{q'Vx} + \mathbf{q'Vq} - \sigma_B^2 \leq 0$. Noting that $\mathbf{q'Vq} = \sigma_B^2$, the expression can be simplified:

$$(\mathbf{x} + 2\mathbf{q})'\mathbf{Vx} \leq 0 \tag{6}$$

This constraint can bind or not. To face this problem, I have to use the features of the ellipse, reported in section 3.1, which describes portfolios with constant TEV in the mean-variance space; indeed, from equation 2 I obtain that $\sigma_P^2 < \sigma_B^2$ if and only if $\Delta_1 < 0$ and $T < \frac{4\Delta_1^2}{d}$. So three cases can be distinguished:

1. $\mu_B \geq \mu_{MV}$ ($\Delta_1 \geq 0$): the ellipse has a positive (or flat) slope and the third constraint is binding (fig.6);

2. $\mu_B < \mu_{MV}$ and $T > \frac{4\Delta_1^2}{d}$: in this case the ellipse has a negative slope, but its center is shifted on the right enough to make the third constraint binding (fig.7);

3. $\mu_B < \mu_{MV}$ and $T \leq \frac{4\Delta_1^2}{d}$: the ellipse has a negative slope and the best portfolio with this TEV covers the commission with his excess return, obtaining a total variance lower (or equal) than the benchmark's variance, so the third constraint do not produce effects (fig.8).

For the first two cases, when the third constraint is binding, this Lagrangian is taken:

$$L = 0.5\mathbf{x'Vx} + \lambda_1(\mathbf{x'1}) + \lambda_2(\mathbf{x'e} - com) + 0.5\lambda_3[(\mathbf{x} + 2\mathbf{q})'\mathbf{Vx}] \tag{7}$$

[10] In this case performance fees, if they exist, could be considered too.

Figure 6: *Frontier with constrained TEV - Case 1.*

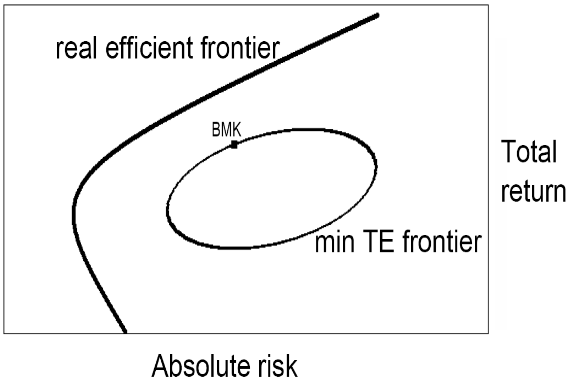


Figure 7: *Frontier with constrained TEV - Case 2.*

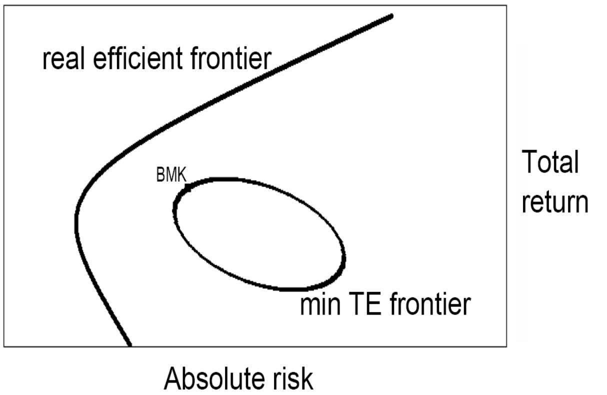
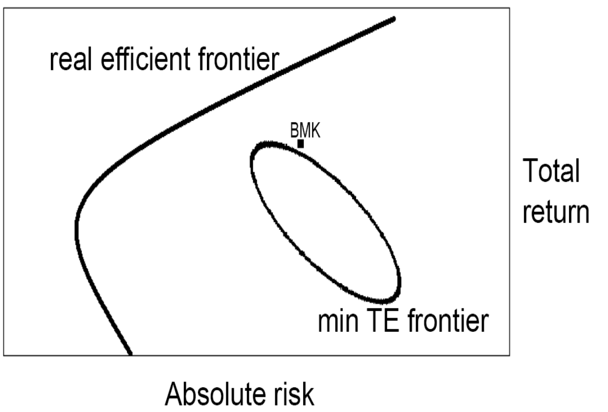


Figure 8: *Frontier with constrained TEV - Case 3.*



Setting the first-order condition $\frac{\partial L}{\partial \mathbf{x}} = 0$, this solution is obtained:

$$\mathbf{x} = \left(\frac{-1}{1+\lambda_3}\right) \cdot \mathbf{V}^{-1} \cdot [\mathbf{1}\lambda_1 + \mathbf{e}\lambda_2 + \lambda_3 \mathbf{Vq}] \tag{8}$$

Substituting equation (8) in the constraints, the values of λs are obtained, so that the three constraints are satisfied. From the first constraint I have:

$$\lambda_1 c + \lambda_2 b + \lambda_3 = 0 \tag{9}$$

where $\mathbf{q'1} = 1$ is used, and c and b are the same as used by Jorion (2003).
From the second constraint I obtain:

$$\lambda_1 \mathbf{1'V}^{-1}\mathbf{e} + \lambda_2 \mathbf{e'V}^{-1}\mathbf{e} + \lambda_3 \mathbf{q'e} = -com \cdot (1+\lambda_3) \tag{10}$$

If $\mathbf{q'e} = \mu_B$ is used, I obtain:

$$\lambda_1 b + \lambda_2 a + \lambda_3 \mu_B + com \cdot (1+\lambda_3) = 0 \tag{11}$$

The third constraint becomes:

$$[\lambda_1 \mathbf{1'V}^{-1} + \lambda_2 \mathbf{e'V}^{-1} + \lambda_3 \mathbf{q'} - 2(1+\lambda_3)\mathbf{q'}]\mathbf{V}(\lambda_1 \mathbf{V}^{-1}\mathbf{1} + \lambda_2 \mathbf{V}^{-1}\mathbf{e} + \lambda_3 \mathbf{q}) = 0 \tag{12}$$

and, with some transformations, I arrive to:

$$\lambda_1^2 c + \lambda_2^2 a - \lambda_3^2 \sigma_B^2 + 2\lambda_1\lambda_2 b - 2\lambda_1 - 2\lambda_2 \mu_B - 2\lambda_3 \sigma_B^2 = 0 \tag{13}$$

The system composed by equation (9), (11) and (13), is then solved to obtain the values of λs. Now eq.(8) is used to obtain the minimum TEV that the portfolio manager has to face if he/she wants to beat the benchmark, $T = \mathbf{x'Vx}$:

$$T = \frac{\lambda_1^2 c + \lambda_2^2 a + \lambda_3^2 \sigma_B^2 + 2\lambda_1\lambda_2 b + 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3\mu_B}{(1+\lambda_3)^2} \tag{14}$$

Substituting the values of λs in the previous equation, the final result (the smallest value) is given by the following expression:

$$T = 2\Delta_2 - \frac{2\Delta_1(\Delta_1 + com)}{d} - \frac{2\Delta_2}{d}\sqrt{\frac{\Delta_1^4 + 2com\Delta_1^3 + com^2\Delta_1^2}{\Delta_2^2} - \frac{2d\Delta_1^2 + 2comd\Delta_1 + com^2 d}{\Delta_2} + d^2} \tag{15}$$

where Δ_1, Δ_2 and d derive from the usual notation used by Jorion (2003) and com is the constant value of commissions.

In the third case, when the third constraint is not binding, this simplier Lagrangian is taken:

$$L = 0.5\mathbf{x'Vx} + \lambda_1(\mathbf{x'1}) + \lambda_2(\mathbf{x'e} - com) \tag{16}$$

Following an analogous proceeding as before, that is setting the partial derivative with respect to x to zero, using this x in the two constraints to obtain the values of λs, and inserting the values of λs in the solution to obtain the minimum TEV, I have this final result:

$$T = \frac{com^2}{d} \tag{17}$$

Combining eq.(17) with the condition $T \leq \frac{4\Delta_1^2}{d}$ (third case), I obtain that equation (17) can be used if

$$com \leq -2\Delta_1 \tag{18}$$

otherwise eq.(15) must be used.

4.2 Analysis and considerations about the two formulas

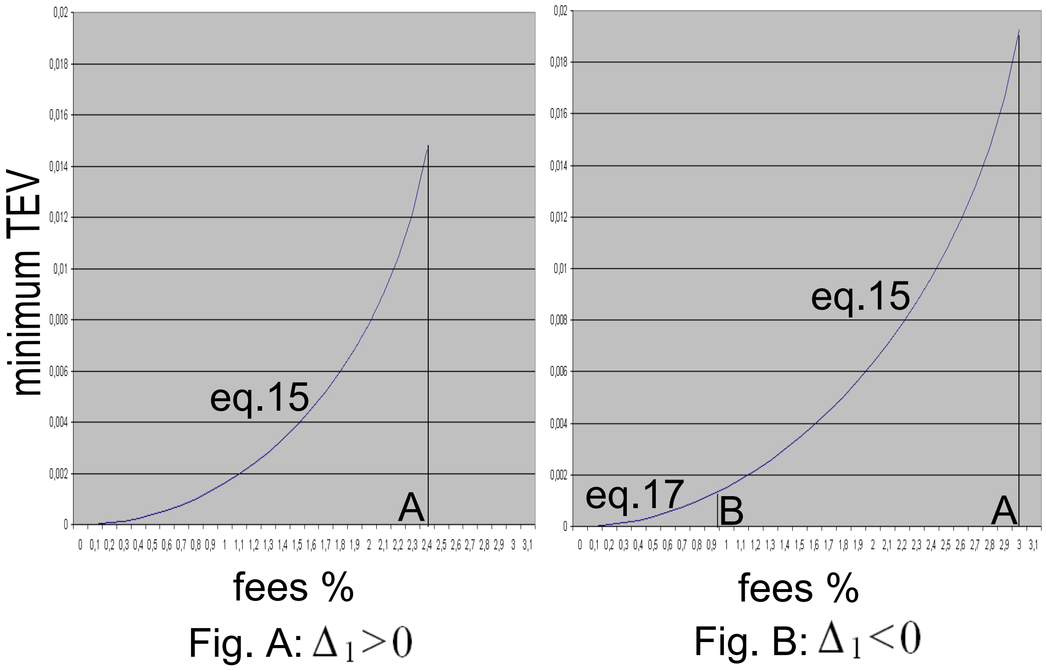
The main driver of the minimum TEV in equation (17) is the amount of commission: with a higher commission, a higher TEV is needed to cover the fee; moreover, the increase of minimum TEV is parabolic (see fig.9), thus, for a high commission level, a small decrease of the fee will strongly decrease the minimum TEV required. This is also true with eq.(15), even if it is not so clear (see appendix A for the analytical demonstration of the positive relationship between fees and minimum TEV).

For all the other components I can say that higher expected returns reduce the minimum TEV in both equations (15) and (17). More generally, in equation (15), a higher value of Δ_1 or a lower value of Δ_2, increases the minimum TEV (see fig.9: comparing figure A with figure B at the same level of commission, the minimum TEV is higher in fig.A where Δ_1 is higher): a more efficient benchmark increases the TEV required. Instead, in eq.(17) the values of benchmark's mean or variance do not modify the TEV value, except when the change makes the third constraint binding and eq.(17) has to be replaced by eq.(15).

I can observe that equation (17) is much simpler than eq.(15), thus, for practical pourposes in a risk management department, eq.(17) could be used as an approximation of eq.(15). Indeed, even if eq.(17) should be used only with an inefficient benchmark ($\mu_B < \mu_{MV}$), it is also true that eq.(15) implies always a minimum TEV equal or higher than the T obtained by eq.(17), so eq.(17) could be seen as a lower bound for TEV, that does not have to be lowered. Moreover, if an asset manager allocates portfolio weights following the ex-ante minimum TEV, he/she should have an ex-post TEV similar to (see Zenti and Pallotta (2002)) or larger than the minimum, because of the stochastic nature of portfolio weights, as shown by Satchell and Hwang (2001) and by the already cited El-Hassan and Kofman (2003). This highlights again that a manager who has an ex-post TEV lower than this minimum value, can not be considered active.

Finally, I outline four important considerations:

Figure 9: *Minimum TEV at different level of commissions. Fig.A:* $\Delta_1 = 0.001$, $\Delta_2 = 0.01$, $d = 0.066$. *Level A of commissions is the maximum level of commission that could be covered:* $d\Delta_2 - \Delta_1^2 = com(com + 2\Delta_1)$. *Fig.B:* $\Delta_1 = -0.005$, $\Delta_2 = 0.01$, $d = 0.066$. *Level A of commissions is the maximum that could be covered,* $d\Delta_2 - \Delta_1^2 = com(com + 2\Delta_1)$, *while at level B* $com = -2\Delta_1$, *so between the origin and level B, the minimum TEV is calculated using eq.(17), while between level B and level A it is calculated with eq.(15).*



1. equation (15) or (17) is a necessary, but not sufficient condition to beat the benchmark. Indeed, an asset manager can take a lot of bets compared to the allocation of the benchmark, but these bets could be wrong. However in this context this is not a problem, indeed if the portfolio manager is not able or lucky[11], the choice of an ETF, for example, is surely superior and the investment trust has no reason to exist;

2. a special case of the above consideration is that the benchmark is on the efficient frontier. In this case the asset manager cannot beat the return of the benchmark if he/she does not want to risk more than his/her referencial index. However, the most rational strategy for an investor that wants the benchmark risk-return profile, is to invest directly in the benchmark (in an ETF that copies it, with a passive strategy) that is already efficient;

3. benchmarks that lie out of the efficient set are avoided, because in these cases the constant TEV frontier is no more an ellipse and all analytical results are not valid; the

[11]Busse *et al.* (2006) find that better-performing portfolios show persistence in their performance, so their managers are probably more able than lucky.

condition, already reported in section 3.1, to be satisfied is $d\Delta_2 - \Delta_1^2 > 0$ so, for example, I do not consider a benchmark grossly inefficient (with a very high negative Δ_1, that is μ_{MV} much higher than μ_B), as the investor could find and pick another index with better mean-variance characteristics;

4. large values of commission are also avoided because, if $com > -2\Delta_1$ (that implies that I am in one of the first two cases), I have to control that $d\Delta_2 - \Delta_1^2 \geq com(com + 2\Delta_1)$, to be able to use eq.(15), otherwise the term under the square root in eq.(15) becomes negative. From the economic perspective it means that it is impossible to have an excess return which covers the fees without increasing the volatility of the portfolio[12]. Thus, combining this condition with the condition of the existence of the ellipse I obtain:

$$d\Delta_2 - \Delta_1^2 > max[0, com(com + 2\Delta_1)] \tag{19}$$

An example can be used to see the relationship between fees and TEV and the limit values for commissions. These features are shown in figure 9. I set $\Delta_2 = 0.01$, $d = 0.066$ and, for fig.A, $\Delta_1 = 0.001$, so $d\Delta_2 - \Delta_1^2 > 0$ and the ellipse exists. With a positive Δ_1 I am in the first case, thus eq.(15) is to be used. It is evident that the minimum TEV required increases with fees and this increase is higher and higher. Level A of commissions is the maximum level that could be covered and it is found applying constraint $d\Delta_2 - \Delta_1^2 = com(com + 2\Delta_1)$. Above level A the asset manager can not cover the fees without increasing the overall volatility of the portfolio return; however, if the asset manager wants to know the minimum TEV to beat the benchmark, accepting a portfolio's variance higher than the benchmark's variance, he/she can use eq.(17).
For fig.B, instead, $\Delta_1 = -0.005$ and, again, $d\Delta_2 - \Delta_1^2 > 0$ so the ellipse exists. With a negative Δ_1 I am in case 2 or 3. Till $com \leq -2\Delta_1$ I am in the third case and the minimum TEV is calculated using eq.(17), while with $com > -2\Delta_1$ I am in the second case and the minimum TEV is calculated using eq.(15). In figure 9 $com = -2\Delta_1$ is at level B, so between the origin and level B eq.(17) is used, while between level B and level A eq.(15) is applied. Level A of commissions is again the maximum that could be covered without increasing the volatility of the portfolio return, calculated as $d\Delta_2 - \Delta_1^2 = com(com + 2\Delta_1)$.

5 An empirical case

A category of funds that could be easily replaced by the correspondent ETFs is the liquidity fund group. Indeed, all these funds show low TEV values and they are often not able to cover the commission fees (that are lower compared to stock funds, but not low enough). For these

[12] In this case eq.(17) can be used to know the minimum TEV required to beat the benchmark, even if with this value I accept a level of the portfolio's variance higher than the benchmark's one.

funds is really difficult to have bets that can move the return of the fund significantly above the benchmark return.
I analyse, for example, the Italian liquidity fund market[13] and, in particular, the TEV of the liquidity fund of the biggest Italian asset management firm. This fund between 2001 and 2007 always underperform its benchmark of a 0.70%-0.80%, because its tracking error is very low and it has a commission of 0.80% (a 0.90% of Total Expense Ratio, TER). Since 2007 the benchmark is 100% the "MTS -ex Banca d'Italia- BOT lordo". The performance in 2007 is analysed: the fund return is 2.7%, while the benchmark return is 3.5% (4% without considering the 12.5% of taxes), the difference is about the value of the commission.
Using our formulas I can understand the causes of the underperformance compared to the benchmark. I calculate the TEV of the fund as the standard deviation of the tracking error between the annualized returns of the fund and of the benchmark, and I obtain a value of 3.3%. I do not know what is the universe of assets in which the fund can invest, but the ten assets with the highest weights in the portfolio are known: they are all Italian government debt titles (BOT, BTP, CCT and CTZ); so I use, as approximation, all the returns data of BOT, BTP, CCT and CTZ that are present in Datastream for 2007.
I control that the benchmark is in the efficient set, indeed $d\Delta_2 - \Delta_1^2 > 0$, so the costant TEV frontier is an ellipse. In theory this fund belongs to the second case ($\Delta_1 < 0$ and $com > -2\Delta_1$), so I have to apply eq.(15). However, I observe that $d\Delta_2 - \Delta_1^2 < com(com + 2\Delta_1)$, so the fees are too high to be covered without increasing the level of portfolio's variance (with our small portfolio composed by BTP-BOT-CCT-CTZ available in Datastream). The maximum level of commission that can be covered with the estimated values of d, Δ_1 and Δ_2 is 0.44%: about a half of the real commission. I can already conclude that the required commission is too high for this kind of fund.
In this case I can only calculate the minimum TEV that can cover commissions accepting an increase in the total variance. Thus I use eq.(17): the value obtained is 3.1%, very near to the real 3.3%. It means that the asset manager has to be very very good to cover all the fee with this minimum use of bets. Indeed with a value of 3.3% for the standard deviation of the TE, the fund can cover at most a fee of 0.85%. Moreover these bets are surely not all good, as I can understand observing the mean of the tracking error which is close to zero (if I add to the portfolio returns the 0.80% of commission).
These values are perhaps inaccurate, because I do not know the real universe of assets avai-

[13] Assogestioni, the association of the Italian investment management industry, classifies liquidity fund as all the funds that have the following characteristics:

- all the portfolio is invested in cash or bonds that must have a rating and this rating have to be at least A2 (Moody's) or A (S&P) or equivalent;
- duration is under 6 months;
- exchange rate risk is not covered.

lable for the portfolio manager. However, this analysis signals strongly that the commission level is set too high and that the activity level is quite low.
This problem is quite common for all liquidity funds. Indeed, if I hypothesize that all Italian liquidity funds have as benchmark the "MTS - ex Banca d'Italia - BOT lordo" and I calculate the TEV for all of them in 2007, I often find similar values; if I hypothesize the same commission level (0.80%) too, some of these funds are also under the minimum level of activity necessary to beat the benchmark even with a higher variance of the overall portfolio (eq.(17)) (I remember that, with our available assets, this level of commission is too high to be beaten without increasing the variance of the portfolio over the variance of the benchmark).

6 Conclusions and possible future works

Asset managers often receive a fee from investors to beat a benchmark, but sometimes they follow it passively and, in this way, they are not able to cover the fees with the excess return obtained over the benchmark. The growth of ETFs forces asset manager to be truly active if they want to remain on the market.

In this paper I analitically derive two formulas, (15) and (17), that fix the minimum tracking error volatility necessary to beat the benchmark index without increasing the overall variance of the portfolio. The risk management department can distinguish if the asset manager is truly active following these equations. Moreover, it can understand if the maximum level of tracking error volatility fixed is too low and makes the benchmark overly difficult to be beaten.

The application of these equations to a real liquidity fund shows that the fees are too high (indeed it is impossible to cover the commission without increasing the overall portfolio variance) and the TEV is quite low. In fact, a lot of funds in the liquidity category show low TEV values that sometimes render the portfolio unable to cover the fees.

From a theoretical point of view, this paper upsets the maximization done by Jorion (2003), generalizing that model with the addition of commissions.

A possible extension of the present work could be the maximum TEV calculated using the risk aversion implicit in the benchmark. The risk aversion implicit in a benchmark can be obtained simply reversing the algorithm used to have the optimal asset weights given a risk aversion parameter, or minimizing the distance of Certainty Equivalent Return (CER) between the optimal and the observed portfolios as done in papers such as DeMiguel *et al.* (2009) or Bucciol and Miniaci (2008). Then the expected TEV for the optimal portfolio can be calculated and this value can be used as maximum TEV, indeed to maximize our investor's utility a TEV higher than this one is not needed. With this method, and with the equations that determine the minimum TEV, the risk management can ask the asset manager to stay in a range of TEV values.

Another possible extension could include higher moments, for example adding a constraint on kurtosis, to force it to be equal or less than the benchmark kurtosis, or a constraint on the whole utility function.

A Appendix: min TEV and fees, analytical derivation

In eq.(17) it is evident that when fees increase, the minimum TEV increases also. This is not immediately evident in eq.(15) also, thus I derive the first derivative in *com* and I show that it is positive.

$$\frac{\partial T}{\partial com} = -\frac{2\Delta_1}{d} - \frac{\Delta_2}{d}\frac{1}{\sqrt{\frac{\Delta_1^4+2com\Delta_1^3+com^2\Delta_1^2}{\Delta_2^2} - \frac{2d\Delta_1^2+2comd\Delta_1+com^2d}{\Delta_2} + d^2}}\cdot$$

$$\cdot\left(\frac{2\Delta_1^3}{\Delta_2^2} + \frac{2com\Delta_1^2}{\Delta_2^2} - \frac{2d\Delta_1}{\Delta_2} - \frac{2comd}{\Delta 2}\right)$$

With a couple of transformations I obtain:

$$\frac{\partial T}{\partial com} = -\frac{2}{d}\left[\Delta_1 + \frac{(\Delta_1 + com)(\frac{\Delta_1^2}{\Delta_2} - d)}{\sqrt{\frac{\Delta_1^4+2com\Delta_1^3+com^2\Delta_1^2}{\Delta_2^2} - \frac{2d\Delta_1^2+2comd\Delta_1+com^2d}{\Delta_2} + d^2}}\right] \tag{20}$$

I know that $-\frac{2}{d}$ is negative because d is positive[14], so I have to show that the expression in brackets is negative:

$$\Delta_1\sqrt{\frac{\Delta_1^4 + 2com\Delta_1^3 + com^2\Delta_1^2}{\Delta_2^2} - \frac{2d\Delta_1^2 + 2comd\Delta_1 + com^2d}{\Delta_2} + d^2} + (\Delta_1 + com)\left(\frac{\Delta_1^2}{\Delta_2} - d\right) < 0 \tag{21}$$

Eq.(15) refers to cases 1 and 2, so I have to distinguish the two cases: in the first Δ_1 is positive, while in the second it is negative.
I can begin with the second case with $\Delta_1 < 0$ and $com > -2\Delta_1$ (otherwise I am in the third case and eq.(17) is to be used, see eq.18):

- $\Delta_1\sqrt{\frac{\Delta_1^4+2com\Delta_1^3+com^2\Delta_1^2}{\Delta_2^2} - \frac{2d\Delta_1^2+2comd\Delta_1+com^2d}{\Delta_2} + d^2} < 0$;
- $(\Delta_1 + com)(\frac{\Delta_1^2}{\Delta_2} - d) < 0$, because $(\Delta_1 + com) > 0$ to fulfil $com > -2\Delta_1$ condition and $(\frac{\Delta_1^2}{\Delta_2} - d) < 0$ to fulfil the condition of existence of the ellipse $d\Delta_2 - \Delta_1^2 > 0$.

Thus equation (21) is surely negative.
In case 1, instead, $\Delta_1 > 0$ so $(\Delta_1 + com)(\frac{\Delta_1^2}{\Delta_2} - d)$ is again negative, but the first part of eq.(21) is positive, thus I have to prove that:

$$\Delta_1^2\left(\sqrt{\frac{\Delta_1^4 + 2com\Delta_1^3 + com^2\Delta_1^2}{\Delta_2^2} - \frac{2d\Delta_1^2 + 2comd\Delta_1 + com^2d}{\Delta_2} + d^2}\right)^2 < \left[(\Delta_1 + com)\left(\frac{\Delta_1^2}{\Delta_2} - d\right)\right]^2$$

[14] d has to be positive because I know, from Jorion (2003), that I have an ellipse if the condition $d\Delta_2 - \Delta_1^2 > 0$ is satisfied, so $d > \frac{\Delta_1^2}{\Delta_2}$ and this is positive because Δ_2 is positive.

Developing the calculus I obtain:

$$d \cdot com(2\Delta_1 + com)\left(\frac{\Delta_1^2}{\Delta_2} - d\right) < 0 \qquad (22)$$

This inequality is surely true because:

- $d \cdot com > 0$;
- $(2\Delta_1 + com) > 0$;
- $(\frac{\Delta_1^2}{\Delta_2} - d) < 0$, as already explained.

So equation (22) is negative and the whole partial derivative (eq.(20)) is positive.
I can conclude that, in any case, the partial derivative of *com* for the minimum TEV is positive: when the fees increase, the minimum TEV required obviuosly increases.

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