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Cournot Duopoly with Capacity Limit Plants

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Abstract

This article considers a Cournot duopoly under an isoelastic demand function and cost functions with built-in capacity limits. The special feature is that each firm is assumed to operate multiple plants, which can be run alone or in combination. Each firm has two plants with different capacity limits, so it has three cost options, the third being to run both plants, dividing the load according to the principle of equal marginal costs. As a consequence, the marginal cost functions come in three disjoint pieces, so the reaction functions, derived on basis of global profit maximization, as well can consist of disjoint pieces. We first analyze the case in which the firms are taken as identical, and then the generic case. It is shown that stable Cournot equilibria may coexist with several other stable cycles. Then we compare the coexistent periodic attractors in terms of the resulting profits. The main property is the non-existence of unstable cycles. This is reflected in a particular bifurcation structure, due to border collision bifurcations, and to particular basin frontiers, related to the discontinuities.

JEL Class.: C61, C62, C72, C73, D21, D24, L13

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Cournot Duopoly with Capacity Limit Plants*

Fabio Tramontana *Laura Gardini* *Tönu Puu*

1 Introduction

In what follows we consider Cournot duopoly where each of the competitors has the option of utilizing several plants. For simplicity suppose each of the firms has only two plants. However, as the firms can either operate each of the plants separately, or both in combination, dividing production between the plants according to the principle of equal marginal costs, it follows that whenever there are two plants, the firms actually have three cost options. Typically, the option chosen depends on output. Suppose we can classify the plants according to their optimal scale of operation; then at a small output the small scale plant will be chosen, with increasing output the choice will shift to the plant appropriate for larger scale production, and eventually both plants will be taken in use, the combination representing the largest scale of all.

Obviously, a constant returns to scale production function will not do. A case suitable for the present discussion is one where capacity limits are explicitly accounted for, as suggested by one of the present authors in several recent publications. See Puu 2005, 2007a,b. To this end one only needs a traditional CES function:

$$q^{-\rho} = k^{-\rho} + l^{-\rho}, \tag{1}$$

assuming $\rho > 0$, and k fixed through an act of investment. Normally, the CES function is formulated with more coefficients, but they can all be dispensed with, using some suitable linear change of scale for output q, and capital and labour inputs k,l. As we are only interested in qualitative properties of the model, which is topologically invariant for all $\rho > 0$, we put $\rho = 1$, so (1) transforms to

$$q = \frac{k_i l}{k_i + l} \tag{2}$$

Next, substituting for $l = k_i q / (k_i - q)$ from (2) in the cost function

$$C_i(q) = \begin{cases} rk_i + wl & \text{if } q > 0 \\ 0 & \text{if } q = 0 \end{cases}$$

* *We wish to thank an anonimous referee for his/her useful comments. we are responsable for all remaining errors.*

we have

$$C_i(q) = \begin{cases} rk_i + w\frac{k_i q}{k_i - q} & \text{if } q > 0 \\ 0 & \text{if } q = 0 \end{cases}$$

The first term represents "fitting up" or "setup" costs that an idle equipment needs when it is put in operation[1]. It is different from the concept of rate of return of the capital, that we do not consider here[2]. The second term represents the variable labour cost. The latter obviously becomes infinite as $q \to k_i$, so k_i is the capacity limit.

Notably, most Cournot dupoly models have a unique equilibrium, which may become unstable and give way to more conplex orbits. Palander 1939, however, proposed several situations where *multistability* could occur. One was the case of a piecewise linear Robinson type demand function where the elasticity increased drastically when price was lowered. Then the marginal revenue function came in discontinuous pieces with jumps, and the corresponding reaction function as well consisted in several pieces. The reaction functions of the dupolists could then also intersect in several points (Cournot equilibria), giving rise to multistability. This case was studied in detail by two of the present authors. See Puu et al. 2002.

Another case elaborated by Palander was where each firm could operate several production plants, some suitable for small scale production (low fixed costs, high marginal costs), other suitable for large scale production (high fixed costs, low marginal costs). In this case it was the marginal cost function that was discontinuous, and, again, multistability could occur. To our knowledge this model was never yet further studied. Like the kinked demand case, Palander based his study on linear functions. We prefer to skip the linear format, using an isoelastic demand function, and non-linear cost functions with built in capacity limits.

We find that reaction functions having one or more discontinuity points, lead to a dynamic behavior which is still associated with stable cycles, but in a situation of *multistability.* This means that the firms must pay particular attention to their states, i.e. in terms of dynamical systems, to their initial conditions, which determine the dynamic evolution towards one situation or a another, which may also be more profitable.

[1]One of the authors once visited a factory for making window glass in France. While the polishing machine for the final processing was in repair, he could see the absurd picture of a huge machine that produced glass sheets which were crushed by workmen with big hammers, and then driven back to the other end of the machine and inserted anew. Obviously, it was too expensive to shut down the machine; it was better to run it until the polishing machine was repaired. Little reflection is needed to convince onself that this is a rather common situation in industry.

[2]In our work "setup" costs (rs_i) are made up by two component: a fixed one (r) and another one, variable with the dimension of the plant ($s_i = f(k_i)$). For example, the reader can think about electricity costs. In particular, we consider the simple case $s_i = k_i$.

The paper is organized as follows: In section 2 we introduce the model, considering the special case in which the two firms are identical (use identical plants in terms of the capacity limits); in section 3 we derive the reaction functions of the duopolists. The model leads to a two-dimensional map M, which reduces to a symmetric map T when the firms are identical, and thus have the same reaction function $\phi(q)$. In section 4 we give the rules to detect all the existing cycles of the duopoly, starting from the cycles of the reaction function $\phi(q)$, showing that all the properties of T can de deduced by the one-dimensional function $\phi(q)$. Local stability analysis is performed, though the cycles never become unstable, because they "disappear" when still stable. In section 5 we compare the profits of the various coexisting attractors, showing several different situations, leading to a "selection problem". In section 6 we deal with the global properties of the two-dimensional map T in presence of multistability: As no unstable cycles exist, we face the problem of the frontiers of the different basins of the coexistent attractors. The basin frontiers are strictly related to the discontinuities of the reaction functions, and we show how to detect them. A second problem is considered, related to changes in the parameter values, and thus to the bifurcations occurring in the cycles of the duopoly. We give numerical evidence that all the bifurcations occurring in the parameter region considered are due to border collision. Section 7 is devoted to the generic case, the duopoly with non-identical firms operating plants with differents capacity limits, and we show that all the main properties discussed in the symmetric case for T persist also for the map M. That is, multistability is a known result, but further, we have no cycle destabilization, just border collision involving stable cycles, and we show that the basin structure is still determined by the discontinuities of the map M, and how to detect these. Some conclusions are drawn in the last section.

2 The model

Suppose that a firm has two plants of limited capacities k_1 and k_2, and $k_2 > k_1$. Obviously, it can also operate both plants at once, dividing the load of production according to the principle of equal marginal costs. We have

$$q = q_1 + q_2$$

Then, minimizing total production cost

$$C\left(q\right) = rk_1 + w\frac{k_1 q_1}{k_1 - q_1} + rk_2 + w\frac{k_2 q_2}{k_2 - q_2} \tag{3}$$

with respect to q_1, q_2 and keeping q fixed, we obtain

$$\begin{cases} q_1 = \frac{k_1}{k_1+k_2} q \\ q_2 = \frac{k_2}{k_1+k_2} q \end{cases},$$

which, substituted back in (3), yield

$$C\left(q\right) = r\left(k_1 + k_2\right) + w\frac{\left(k_1 + k_2\right) q}{\left(k_1 + k_2\right) - q}.$$

Denoting this function $C_3(q)$, and defining $k_3 = k_1 + k_2$, we have automatically a "third plant" to operate, the options are

$$C_i(q) = rk_i + w\frac{k_i q}{k_i - q} i = 1, 2, 3 \tag{4}$$

with $k_3 = k_1 + k_2 > k_2 > k_1$. (Similarly, with three plants the firm will have seven different options to choose among, and so forth.) The choice between the presently three options depends on total costs, as we will see. Graphically the situation is shown in Fig.1, where q^* denotes the output for which the costs for the two plants operated alone break even.

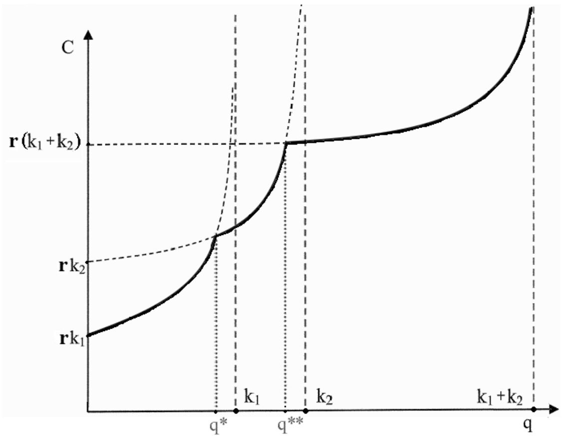

Figure 1: Total Costs

Solving the equation $C_1(q^*) = C_2(q^*)$, i.e., $rk_1 + w\frac{k_1 q^*}{k_1 - q^*} = rk_2 + w\frac{k_2 q^*}{k_2 - q^*}$ we get:

$$q^* = \frac{r(k_1 + k_2) - \sqrt{r^2(k_1 + k_2)^2 - 4(r - w)rk_1 k_2}}{2(r - w)}. \tag{5}$$

Similarly q^{**} denotes the output for which the cost for using the larger capacity plant alone breaks even with using both in combination. Solving the equation $C_2(q^{**}) = C_3(q^{**})$, i.e., $rk_2 + w\frac{k_2 q^{**}}{k_2 - q^{**}} = r(k_1 + k_2) + w\frac{(k_1 + k_2)q^{**}}{k_1 + k_2 - q^{**}}$, we get

$$q^{**} = \frac{r(k_1 + 2k_2) - \sqrt{r^2(k_1 + 2k_2)^2 - 4(r - w)r(k_1 + k_2)k_2}}{2(r - w)}. \tag{6}$$

Let us define the following intervals for the quantity of product: $J_1 =]0, q^*[$, $J_2 =]q^*, q^{**}[$, $J_3 =]q^{**}, (k_1 + k_2)[$. If the optimal choice q belongs to J_i then $C_i(q)$ applies. The cost function of the model is given by

$$C(q) = \left\{ C_i(q) = rk_i + w\frac{k_i q}{k_i - q} \quad \text{if } q \in J_i, \quad i \in \{1, 2, 3\} \right. \tag{7}$$

The three different total cost functions give rise to three different marginal cost functions:

$$MC(q) = \Big\{ MC_i(q) = \tfrac{dC_i(q)}{dq} = w\tfrac{k_i^2}{(k_i-q)^2} \quad \text{if } q \in J_i, \quad i \in \{1,2,3\} \tag{8}$$

Graphically they are shown in Fig.2. Clearly at q^* and q^{**} we have discontinuity points for marginal cost MC. Assuming an isoelastic demand function

$$p = \frac{1}{Q_{-1} + q},$$

where Q_{-1} denotes residual market supply (not under control of the firm itself), total revenue becomes:

$$R(q) = \frac{q}{Q_{-1} + q}, \tag{9}$$

from which marginal revenue is:

$$MR = \frac{Q_{-1}}{(Q_{-1} + q)^2} \tag{10}$$

It is a decreasing function with respect to q, and an example is also added in Fig.2.

We look for the maximum profit (so as to obtain the reaction function of the firm). From the function $\Pi(q) = R(q) - C(q)$, in order to satisfy the first order conditions we have to equate marginal revenue to marginal cost, which is a discontinuous function. Depending on the parameters of the model we may have one, two or even three intersections (an example of which is shown in Fig. 2). From the second order conditions it is easy to see that *they are all local maxima*. The number of intersection points of MR and MC depends on the position of the marginal revenue curve, which may also change over time periods, as it is a function of Q_{-1}.
Analytically the intersection points are obtained solving

$$\frac{Q_{-1}}{(Q_{-1} + q)^2} = w\frac{k_i^2}{(k_i - q)^2} \tag{11}$$

and when we have an intersection belonging to J_i then the value of the intersection point is:

$$q_i^* = k_i \frac{\sqrt{\frac{Q_{-1}}{w}} - Q_{-1}}{k_i + \sqrt{\frac{Q_{-1}}{w}}} \quad \text{if } q_i^* \in J_i \tag{12}$$

Considering several competitors, the simplest case is duopoly, either with different plant capacities, or, even simpler, identical firms. We first study this latter case, and then the more general.

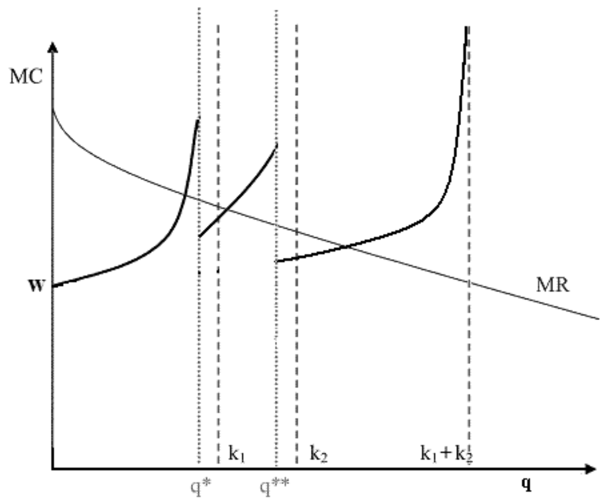

Figure 2: Marginal Cost and Marginal Revenue.

As we always keep to duopoly, we switch to a simpler notation, letting x and y denote the outputs produced by the duopolists in time period t. Focusing the first producer, its best reply for the next time period $(t+1)$ is

$$x_i^*(y) = k_{x,i}\frac{\sqrt{\frac{y}{w}} - y}{k_{x,i} + \sqrt{\frac{y}{w}}} \quad \text{if } 0 < y < 1/w \quad i = 1, 2, 3 \tag{13}$$

$$x_4^*(y) = 0 \quad \text{if } y \geq 1/w \tag{14}$$

where $k_{x,i}$ denotes the capacity limit of the i:th plant for the first dupolist (and similarly $k_{y,j}$ denotes the capacity limit of the j:th plant for the second dupolist). By construction, as $k_{x,1} < k_{x,2} < k_{x,3} = (k_{x,1} + k_{x,2})$ and $q_x^* < q_x^{**}$, we also have $x_1^* < x_2^* < x_3^*$ and at each time period (or iteration) the choice among the three values is the one which gives the maximum profit.

3 The reaction functions

As the decision for production in time period $(t+1)$ is taken on the basis of the maximum profit, we have to compare the profits in the three cases, given by:

$$\Pi_{x,i}(y) = \frac{x_i^*(y)}{y + x_i^*(y)} - rk_{x,i} - w\frac{k_{x,i}x_i^*(y)}{k_{x,i} - x_i^*(y)} \quad i = 1, 2, 3$$

Denoting maximum profits of the first firm Π_x, i.e., $\Pi_x = \max\{\Pi_{x,i}(y), i = 1, 2, 3\}$, we have

$$x' = \begin{cases} x_i^*(y) \quad \text{which gives} \quad \Pi_x & \text{if } 0 < y < 1/w \\ 0 & \text{if } \ y \geq 1/w \end{cases} \tag{15}$$

Similarly, for the other competitor we have the best reply:

$$\begin{array}{llr} y_j^*(x) = k_{y,j}\dfrac{\sqrt{\frac{x}{w}} - x}{k_{y,j} + \sqrt{\frac{x}{w}}} & \text{if } 0 < x < 1/w \ \text{ j=1,2,3} & \\ y_4^*(y) = \qquad 0 & \text{if} \quad x \geq 1/w & \end{array} \tag{16}$$

so comparing these profits:

$$\Pi_{y,j}(x) = \frac{y_j^*(x)}{x + y_j^*(x)} - rk_{y,j} - w\frac{k_{y,j}y_j^*(x)}{k_{y,j} - y_j^*(x)} \quad j = 1, 2, 3 \tag{17}$$

we obtain:

$$y' = \begin{cases} y_j^*(x) & \text{which gives } \ \Pi_y \quad \text{if } 0 < x < 1/w \\ 0 & \qquad\qquad\qquad\quad \text{if} \quad x \geq 1/w \end{cases} \tag{18}$$

where $\Pi_y = \max\{\Pi_{y,j}(y),\ j = 1, 2, 3\}$ is the maximum profit for the second duopolist.

In the simple duopoly case (15) and (18) are the reaction functions. It is important to stress that the reaction functions may have *discontinuities.* A discontinuity happens when, increasing or decreasing the output of the competitor, a different plant operation becomes more profitable than the previous. All reaction branches defined above can hence contribute to the final reaction curve.

The competitors, of course, can be assumed to face the same market prices for capital and labour, but there is no reason to assume that their capacity limits $k_{x,i}$, $k_{y,j}$ are the same. The competitors may not even need to have an equal number of plants to operate, and we extend our analysis to the generic case in section 7.

Even if the reaction function corresponding to a particular plant operation is part of the final reaction function, this does not mean that the firm will actually use that plant, because it depends on the dynamic behavior, i.e., the values corresponding to the related branch may not be obtained dynamically. Anyhow, the final reaction function is the important one, in order to understand to which branch the attractors of the dynamic system (formed by eq.s 15 and 18) belong.

In the next section we analyze the coexistence of cycles of the system.

4 Emergence of multistability

In duopoly games, we have interdependence between firms, i.e., the optimal output of a firm is a function of the expected production of the competitor: $x = \phi(y^e)$. Every time period the firms form their expectations, and when they are static, $y_{t+1}^e = y_t$, the problem admits a unique solution for every choice by the competitor, than we obtain a dynamic system of this kind:

$$M : \begin{cases} x_{t+1} = \phi_1(y_t) \\ y_{t+1} = \phi_2(x_t) \end{cases} \tag{19}$$

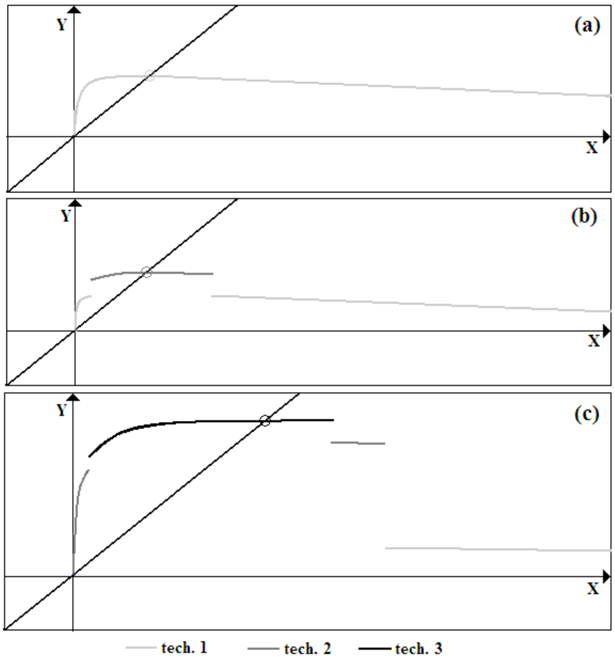

Figure 3: Reaction Curves.
In (a) the final reaction curve corresponds to a unique plant operation, and is obtained with the following values of the parameters: $k_1 = 0.397$, $k_2 = 0.518$, $r = 0.78$, $w = 0.15$. In (b) the final reaction curve is formed by branches of two different reaction curves, and is obtained with the following values: $k_1 = 0.122$, $k_2 = 0.233$, $r = 0.65$, $w = 0.559$. In (c) the final reaction curve includes branches from all the three reaction curves, and is obtained with the following values: $k_1 = 0.1$, $k_2 = 0.322$, $r = 0.452$, $w = 0.134$. The three basic curves are drawn in different gray tonalities.

The time evolution of this class of maps (having the second iterate which is a separate system in x and y) has been studied in Bischi et al. 2000 (see also Lenci et al. 1997) in which the authors show that, *in the case of continuous reaction functions,* multistability emerges easily and they demonstrate that a complete understanding of all the cycles of the two-dimensional map and their stability properties are given by one of these one-dimensional maps (correlated to each other):

$$F(x) = \phi_1 \circ \phi_2(x) \qquad ; \qquad G(y) = \phi_2 \circ \phi_1(y) \tag{20}$$

In our case, the model is generally described by discontinuous reaction functions. However, it is easy to prove that several properties stated in Bischi et al. 2000 also holds in the discontinuous case, specially those associated with the existence and coexistence of cycles, while this is not true for the bifurcations related with the cycles and the structure of the basin boundaries.

The system formed by (15) and (18), where the parameters $k_{x,i}$ and $k_{y,i}$ are the same for the two firms (i.e. identical firms, or symmetric case) corresponds to the particular case of (19) with $\phi_1 = \phi_2 = \phi$, so that in the symmetric case we are interested in a two-dimensional map having the following structure:

$$T : \begin{cases} x_{t+1} = \phi(y_t) \\ y_{t+1} = \phi(x_t) \end{cases} \tag{21}$$

Thus, in the case of identical firms we can reduce the study of T to that of the one-dimensional map $\phi(x)$. Indeed in such a case we have $F = G$ and $F(x) = \phi^2(x)$, and the properties of $\phi^2(x)$ only depend on those of $\phi(x)$. As already stated above, some results associated with this class of maps (with separate second iterate) are not limited to the case of *continuous* reaction function, but also extend to the case in which the reaction function $\phi(x)$ is piecewise smooth, with one or more *discontinuity points*. Let us start with some simple results for the symmetric case. Due to the unique reaction functions, we have that the diagonal of the phase space is mapped into itself, and the restriction of the two-dimensional map T on this invariant set reduces to the one-dimensional map $r(x)$, moreover, the usual property of symmetric systems holds, as stated in the following:

Proposition 1 *In a duopoly game with identical reaction functions ($\phi_1 = \phi_2 = \phi$) then:*

- *(a) the diagonal Δ (the straight line $x = y$) is a trapping set (i.e. $T(\Delta) \subseteq \Delta$),*
- *(b) any invariant set I of the phase plane (i.e. such that $T(I) = I$), either is symmetric with respect to Δ, or the symmetric one is also invariant.*

Proof. The proof of (a) is immediate: let $(x, x) \in \Delta$ then $T(x, x) = (\phi(x), \phi(x)) \in \Delta$. To prove (b) let us denote by S the symmetric operator such that $S(a, b) = (b, a)$, then, from the definition of T, we have $S(T(x, y)) = T(S(x, y))$. Now let I be an invariant set of T, so that $T(I) = I$, then $S(T(I)) = T(S(I)) = S(I)$ holds. It follows that either $S(I) = I$ (i.e. I is invariant) or $I' = S(I)$ is invariant (being $I' = T(I')$). ■

In Bischi et al. 2000 it is demonstrated that to each n-cycle of F : $\{x_1, ..., x_n\}$ there corresponds a *conjugate* n-cycle of G given by: $\{y_1, ..., y_n\} = \{\phi_2(x_1), ..., \phi_2(x_n)\}$ and if we consider all the periodic points of F (and their conjugates) then the Cartesian product $\{x_1, ..., x_n\} \times \{y_1, ..., y_n\}$ gives all the periodic points of the map in (19). This property, which continues to hold also in the discontinuous case, i.e. for the map M in (19), for the two-dimensional map T in (21) becomes:

Proposition 2 *Let $\{x_i\}$ be the set of all the periodic points of the map $\phi(x)$, then the set of conjugate points is the same set $\{x_i\}$, and the points of the Cartesian product $\{x_1, ..., x_n\} \times \{x_1, ..., x_n\}$ give all the periodic points of the map T in (21).*

These two Properties are very useful because we can obtain the coordinates of the periodic points of T (belonging to the Cartesian product) considering only the dynamics on the diagonal Δ (where the map reduces to $\phi(x)$). Summarizing, we only need to study $\phi(x)$ to know the number and the coordinates of all the cycles of our two-dimensional map T, knowing that:

$$\phi(x) = \begin{cases} k_i \dfrac{\sqrt{\frac{x}{w}} \ - \ x}{k_i \ + \ \sqrt{\frac{x}{w}}} & if \quad 0 < x < 1/w \\ 0 & if \qquad x \geq 1/w \end{cases} \tag{22}$$

where i is so chosen that the profit is a maximum. Thus, as we have seen, $\phi(x)$ may be formed by several disjoint pieces of the reaction functions of the three production plant choices.

4.1 From $\phi(x)$ to T

In this subsection we describe the properties which allow to study the map T by analyzing only the one-dimensional map $x' = \phi(x)$, where $\phi(x)$ is defined in (22). Let us consider the action of the map as:

$$T(x, y) = (\phi(y), \phi(x))$$

and let $\{x_1, x_2, ..., x_n\}$ be a cycle of $\phi(x)$ of first period n ($x_{i+1} = \phi(x_i)$ and $\phi^n(x_i) = x_i$ for $i = 1, ..., n$), and consider the points of the Cartesian product $\{x_1, ..., x_n\} \times \{x_1, ..., x_n\}$. Then one can immediately compute the iterates by T which are as follows:

$$T^k(x_i, x_j) = \begin{cases} (\phi^k(x_j), \phi^k(x_i)) & if \ \ k \ \ is \ \ odd \\ (\phi^k(x_i), \phi^k(x_j)) & if \ \ k \ \ is \ \ even \end{cases} \tag{23}$$

If $i = j$ we have a point on Δ and thus the first integer giving a cycle is $k = n$; (and we get the $n-$cycle on Δ), while for $i \neq j$ we have a point (x_i, x_j) external to Δ, and the first integer giving a cycle depends on the period n. If n is odd then the first integer giving a periodic point in (23) is $k = 2n$ so that (x_i, x_j) belongs to a cycle of T external to Δ of first period $2n$. Such distinct cycles must equal $(n^2 - n)/(2n) = (n-1)/2$ in number. If n is even, then the first integer giving a periodic point in (23) is $k = n$ so that (x_i, x_j) is certainly periodic of period n, external to Δ, and at most $(n^2 - n)/n = (n-1)$ distinct cycles of period n can exist. However the prime period may be less than n. This happens only when n is even and $n/2$ is odd, and the periodic points belonging to two distinct cycles of period $n/2$ are $(x_i, x_{i+n/2})$ and $(x_{i+n/2}, x_i)$. We have thus proved the following proposition which classifies the cycles which we shall call *singly-generated* because their existence for the map T is a direct consequence of the existence of one cycle for $\phi(x)$:

Proposition 3 (singly-generated cycles) *Let* $\{x_1, x_2, ..., x_n\}$ *be a cycle of* $\phi(x)$ *of first period* $n \geq 1$,

- if n is odd then T has:

  (a) one cycle of the period n (on Δ)

  (b) $(n-1)/2$ cycles of period $2n$ (external to Δ)

- if n is even and $n/2$ is also even then T has:

  n cycles of period n (one on Δ and $(n-1)$ external to Δ)

- if n is even and $n/2$ is odd then T has:

  (a) 2 cycles of period $n/2$ (external to Δ)

  (b) $(n-1)$ cycles of period n (one of which on Δ)

Now let us consider the case in which two or more cycles of $\phi(x)$, of any pair of periods, coexist. Without loss of generality let us consider a cycle of period n, $n \geq 1$, say $\{x_1, ..., x_n\}$, and a cycle of period m, $m \geq 1$, say $\{y_1, ..., y_m\}$. Then it is clear (from Proposition 3) that $n \times n$ points of the type (x_i, x_j) belong to *singly-generated* cycles of T, and as well $m \times m$ points of the type (y_i, y_j) belong to *singly-generated* cycles of T, but in the Cartesian product $\{x_1, ..., x_n, y_1, ..., y_m\} \times \{x_1, ..., x_n, y_1, ..., y_m\}$ we also have other periodic points. Such points belong to cycles which we shall call *doubly-generated,* because each point of the cycles has the coordinates belonging to two different cycles of $\phi(x)$, so that their existence for the map T is a direct consequence of the existence of a *pair of cycles of* $\phi(x)$. To see this let us consider the iterates by T which are as follows:

$$T^k(x_i, y_j) = \begin{cases} (\phi^k(y_j), \phi^k(x_i)) & if \quad k \quad is \quad odd \\ (\phi^k(x_i), \phi^k(y_j)) & if \quad k \quad is \quad even \end{cases} \tag{24}$$

and let us define

$$S = \text{lcm}(n, m)$$

where "lcm" stands for "least common multiple". Then it is clear that when S is odd (which can occur only when both n and m are odd), then the least integer giving a periodic point in (24) is $k = 2S$, and we get a cycle of T of period $2S$. Such cycles may be $2(n \cdot m)/(2S) = n \cdot m/S$ in number. When S is even (which occurs when n or/and m are even), then the least integer giving a periodic point in (24) is $k = S$, so that we get a cycle of T of period S, and such cycles may be in number $2(n \cdot m)/S$. We have so proved the following proposition which classifies the *doubly-generated* cycles associated with a pair of cycles of $\phi(x)$:

Proposition 4 *(*doubly-generated cycles) *Let $\{x_1, ..., x_n\}$ be a cycle of $\phi(x)$ of first period $n \geq 1$, and $\{y_1, ..., y_m\}$ a cycle of $\phi(x)$ of first period $m \geq 1$, and let S be the least common multiple between n and m, then the cycles of T of type* doubly-generated *are as follows:*

- if n and m are odd then T has $\frac{n \cdot m}{S}$ cycles of period $2S$

- if n or/and m are even then T has $\frac{2n \cdot m}{S}$ cycles of period S.

Particular attention must be paid to the *fixed points* of T. Clearly (x^*, y^*) is a fixed point of T if and only if $T(x^*, y^*) = (\phi(y^*), \phi(x^*)) = (x^*, y^*)$. If x^* is a fixed point of $\phi(x)$, then (x^*, x^*) is a fixed point of T (of singly-generated type) belonging to the diagonal Δ. When we have two fixed points of T, say x^* and y^* then T has two fixed points (of singly-generated type, (x^*, x^*) and (y^*, y^*)) belonging to the diagonal Δ. Further, by Proposition 4 a $2-$cycle of T of doubly-generated type external to Δ is get (with periodic points $\{(x^*, y^*), (y^*, x^*)\}$) . When x_1 and x_2 are the points of a $2-$cycle of $\phi(x)$, then the related singly-generated cycles of T are (by Proposition 3) a $2-$cycle $\{(x_1, x_1), (x_2, x_2)\}$ on Δ, and two fixed points external to Δ (given by $X^* = (x_1, x_2)$ and $Y^* = (x_2, x_1)$). It is worth noting that in the symmetric case it is not possible to get only one fixed point of T external to the diagonal (as otherwise also the symmetric one must exist, by Proposition 1), while for the generic map M in (19) this is possible. In fact, a fixed point must be an intersection point of the two reaction functions, so that it is characterized by $x^* = \phi_1(y^*)$ and $y^* = \phi_2(x^*)$, and we may also have a single fixed point of M not belonging to the diagonal.

We have not yet said a word about the stability/instability of the cycles of T. In the case in which the reaction function $\phi(x)$ *is piecewise smooth, with one or several discontinuity points*, we shall assume that the considered cycles of $\phi(x)$ have periodic points in which the function is differentiable, that is, for now we are not considering the bifurcations related to the appearance/disappearance of such cycles, which in our case mainly occurs by *border-collision bifurcations.* Here we are only interested in their local stability, once they exist. As we have shown above, any cycle of T is related to one (if singly-generated) or two (if doubly-generated) cycles of $\phi(x)$, and it is very easy to see, considering the Jacobian matrix of T, that the following proposition holds:

Let $X = \{x_1, ..., x_n\}$ be a cycle of $\phi(x)$ of first period $n \geq 1$, and $Y = \{y_1, ..., y_m\}$ a cycle of $r(x)$ of first period $m \geq 1$, then

- if X is asymptotically stable (resp. unstable) for $\phi(x)$ with eigenvalue λ, $|\lambda| < 1$ (resp. $|\lambda| > 1$) then all the *singly-generated* cycles associated with X are asymptotically stable (unstable), *star nodes* for T with eigenvalues $\zeta_1 = \zeta_2 = \lambda$;

- if X and Y are both asymptotically stable for $\phi(x)$, with eigenvalues λ and μ ($|\lambda| < 1$, $|\mu| < 1$), then all the *doubly-generated* cycles associated with X and Y are asymptotically *stable nodes* for T with eigenvalues $\zeta_1 = \lambda$, $\zeta_2 = \mu$;

- if X or Y is unstable for $\phi(x)$, with eigenvalues λ and μ, $|\lambda| < 1$ and $|\mu| > 1$, then all the *doubly-generated* cycles associated with X and Y are unstable for T, of *saddle* type, with eigenvalues $\zeta_1 = \lambda$, $\zeta_2 = \mu$;

- if X and Y are both unstable for $\phi(x)$, with eigenvalues λ and μ ($|\lambda| > 1$, $|\mu| > 1$), then all the *doubly-generated* cycles associated with X and Y are *unstable nodes* for T, with eigenvalues $\zeta_1 = \lambda$, $\zeta_2 = \mu$.

As an example, consider the map T with parameters: $k_1 = 0.2$, $k_2 = 0.54$, $r = 0.518$, $w = 0.15$. The one-dimensional map has a stable $3-$cycle with coordinates $\{x_1, x_2, x_3\}$ which corresponds to a $3-$cycle on the diagonal for T (Fig. 4). The Cartesian product $\{x_1, x_2, x_3\} \times \{x_1, x_2, x_3\}$ is formed by 9 points, and the related cycle with points outside the diagonal Δ is obtained from Proposition 3 so that we find that they must form a $6-$cycle (external to Δ in Fig.4).

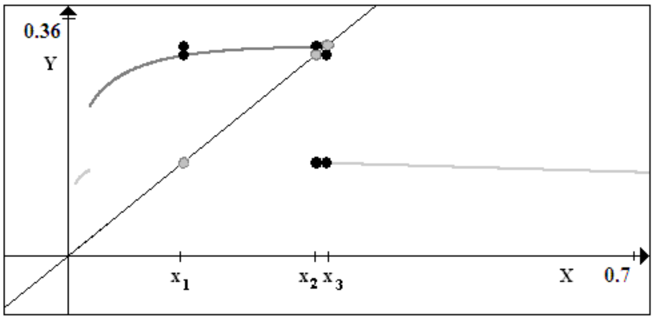

Figure 4: Two coexisting attractors of the map T.
This is the set of parameters used: $k_1 = 0.2$, $k_2 = 0.54$, $r = 0.518$, $w = 0.15$: A stable $3-$cycle on the diagonal Δ and a symmetric $6-$cycle external to Δ.

5 Profits of the firms and the market: Interesting situations

In this section we are going to analyze the profits which are obtained by the firms and the market, when the dynamics have reached the asymptotic states. During the iteration process the first firm "assumes" that the other one keeps the quantity of the previous period (say y_t) and decides its own production (say x_{t+1}) in a process which selects maximum profit. Similarly for the second firm. But we did not compute, dynamically, the actual profits of the firms. So let us define *private profit* $P_x(A)$ of the first firm in a point $A = (x, y)$, i.e., the profit of the firm when the outputs of the duopolists are the coordinates of the point A. We have

$$P_x(A) = \frac{x}{x+y} - C_x \quad \text{where} \quad C_x = rk_{x,i} - w\frac{k_{x,i}x}{k_{x,i} - x} \quad \text{for } i \in \{1, 2, 3\}\,, \tag{25}$$

and similarly for the second firm:

$$P_y(A) = \frac{y}{x+y} - C_y \quad \text{where} \quad C_y = rk_{y,j} - w\frac{k_{y,j}y}{k_{y,j} - y} \quad \text{for } j \in \{1, 2, 3\}\,. \tag{26}$$

The parameters $k_{x,i}$ and $k_{y,j}$ depend on the branches of the reaction functions involved, which may also be different. We also define the *private profit*

$P_x^{(m)}(A)$ as the sum of the private profits of the first firm in the m periods of the trajectory starting from A. Similarly for $P_y^{(m)}(A)$. Finally, we are interested in the *total profit* $TP(A)$ in a point $A = (x, y)$, which is the *sum of the private profits of both the firms* in A, that is, for some suitable $i, j \in \{1, 2, 3\}$:

$$TP(A) = P_x(A) + P_y(A) = 1 - (C_x + C_y) = \tag{27}$$

$$= 1 - r(k_{x,i} + k_{y,j}) - w\left(\frac{k_{x,i}x}{k_{x,i} - x} + \frac{k_{y,j}y}{k_{y,j} - y}\right) \tag{28}$$

and the *total profit* $TP^{(m)}(A)$ is the sum of the total profits in the m periods of the trajectory starting from A.

Clearly in a point on Δ we have $P_x(A) = P_y(A)$ because $x = y$ and $k_{x,i} = k_{y,i}$, so that we also have $P_x^{(n)}(A) = P_y^{(n)}(A) = \frac{1}{2}TP^{(n)}(A)$ for each $n > 0$.

If we consider a point $B = (x, y)$ which does not belong to the diagonal, its private profits $P_x(B)$ and $P_y(B)$ are different, and if we take the symmetric point $B' = (y, x)$ we clearly have $P_x(B') = P_y(B)$ and $P_y(B') = P_x(B)$, because in B the first (resp. second) firm produces the same quantity (and uses the same plant capacities) as the second (resp. first) firm in B'. It follows that the total profit in symmetric points is the same:

$$TP(B) = P_x(B) + P_y(B) = 1 - (C_x + C_y) = \tag{29}$$

$$= P_y(B') + P_x(B') = TP(B') \tag{30}$$

Moreover, in symmetric points both firms have the same sum of private profits, which also is the same value as the total profit:

$$P_x(B) + P_x(B') = P_y(B') + P_y(B) = 1 - (C_x + C_y) \tag{31}$$

For example, from an economic point of view it is quite interesting to notice that in the case shown in Fig. 4 the *private profit of both firms after 6 periods* is the same in both the different attractors. In fact, let $A = (x_1, x_1)$ be a point of the 3-cycle on Δ. Then we know that $P_x^{(n)}(A) = P_y^{(n)}(A) = \frac{1}{2}TP^{(n)}(A)$ for each $n > 0$. Further let $B = (x_1, x_2)$ be a point of the 6-cycle outside Δ. The 6-cycle is formed by 3 pairs of symmetric points, in each pair of which the sum of the private profits is the same, and after 6 periods every point is visited once, so that the total profit after 6 periods is equidistributed: $P_x^{(6)}(B) = P_y^{(6)}(B) = \frac{1}{2}TP^{(6)}(B)$. But it can be proved that these values are equal, i.e. $TP^{(6)}(A) = TP^{(6)}(B)$ and $TP^{(3)}(A) = \frac{1}{2}TP^{(6)}(B)$, so that $P_x^{(6)}(B) = P_y^{(6)}(B) = P_x^{(3)}(A) + P_y^{(3)}(A)$. This comes from the following:

Proposition 5 *Let $X = \{x_1, ..., x_n\}$ be a cycle of $\phi(x)$ of first period $n > 1$, and $s = n(n-1)$, then*

- *the sum of the total profits of the s periodic points of T singly generated from X (and external to Δ) is equal to $TP^{(s)}(x_1, x_1)$, i.e. $(n-1)$ times the total profit $TP^{(n)}$ on the n-cycle X on the diagonal;*

- *the sum of the total profits of each firm (first and/or second) of the s periodic points of T singly generated from X (and external to Δ) is equal to* $P_x^{(s)}(x_1, x_1) = P_y^{(s)}(x_1, x_1) = \frac{1}{2}TP^{(s)}(x_1, x_1)$.

Proof. As already noted above, for $A \in \Delta$ we have $P_x^{(k)}(A) = P_y^{(k)}(A) = \frac{1}{2}TP^{(k)}(A)$ for each $k > 0$, and from $TP^{(n)}(A) = n - 2\sum_{i=1}^{n} C_{x_i}$ we have $TP^{(s)}(A) = (n-1)n - (n-1)2\sum_{i=1}^{n} C_{x_i}$.

Now we know (from Proposition 1) that the periodic points external to the diagonal are symmetric with respect to Δ, so that we have $s/2 = n(n-1)/2$ pairs of points external to Δ, and for each pair, say $B = (x_i, x_j)$ and $B' = (x_j, x_i)$, we have $P_x(B) + P_x(B') = P_y(B') + P_y(B) = 1 - (C_{x_i} + C_{x_j})$, so that, summing up for all pairs, and taking in account that we have only n distinct values, we obtain $P_x(total) = P_y(total) = \frac{n(n-1)}{2} - \frac{(n-1)}{2}(\sum_{i=1}^{n} C_{x_i} + \sum_{j=1}^{n} C_{x_j}) = \frac{n(n-1)}{2} - (n-1)\sum_{i=1}^{n} C_{x_i}$ for the sum of the total profits of the singly generated periodic points. As the total profit is the sum of the two values, we get the same as $TP^{(s)}(A)$.

Moreover, whichever is the period n (odd or even), from the explicit formulas written above we have also obtained that $P_x(total) = P_y(total) = \frac{n(n-1)}{2} - (n-1)\sum_{i=1}^{n} C_{x_i} = \frac{1}{2}TP^{(s)}(A)$,which completes the proof. ■

Coming back to the case shown in Fig. 4, with a 3-cycle on Δ and a 6-cycle outside, after 6 periods we get the same total profit, both starting from a periodic point on Δ and starting from a periodic point outside the diagonal. But, in general, a situation in which the total profit is equal in all the coexistent cycles is quite rare. Indeed the case $n = 3$ is particular, because it is the unique period for which we have only one singly generated cycle outside the diagonal. In general, outside Δ there are more coexisting singly-generated cycles, and total profit on different cycles differs (although globally Proposition 5 above holds).

Moreover, we may have coexisting cycles on the diagonal (and this may also be considered as generic in our model). In such cases, when m cycles (with $m > 1$) belong to Δ, outside the diagonal we have not only singly-generated cycles, but also doubly-generated cycles, and we can find two kinds of interesting situations in terms of ecomomics:

- *Different profits for the firms*: occur when we can find at least one cycle of period k say, with periodic point (x, y), in which the cumulative profit of one firm ($P_x^{(k)}(x, y)$ or $P_y^{(k)}(x, y)$) is higher than the cumulative profit of other firm (so that, for example, if $P_x^{(k)}(x, y) > P_y^{(k)}(x, y)$ then $P_x^{(kn)}(x, y) > P_y^{(kn)}(x, y)$ for each $n \geq 1$).

- *Different profits for the market*: occur when among the coexisting cycles there exists one cycle (at least), with periodic point (x, y), having the highest total profit $TP^{(m)}(A)$, where m is the least common multiple of all the periods of the cycles of T.

To illustrate the different kinds of behaviors, we propose three examples. The first (in the next subsection) shows a situation of *different profits for the firms*. In subsection 5.2, the second example seems, at first sight, similar to the first, whereas it represents a case of *different profits for the market*. The third and last example shows a more complicated (and generic) case of *different profits for the market*, in which 10 different attractors coexist.

5.1 Different profits for the firms

Let us consider the situation with parameters: $k_1 = 0.024$, $k_2 = 0.562$, $r = 0.78$, $w = 0.15$. In this example, the one-dimensional map $x' = \phi(x)$ has a stable 2-cycle $\{x_1, x_2\}$. Using the rules given in the previous section (Proposition 3) we know that the map T has an attracting 2-cycle on the diagonal, and two stable and symmetric *singly-generated* fixed points outside Δ: $A = (x_1, x_2)$ and $B = A' = (x_2, x_1)$ (both shown in Fig. 5).

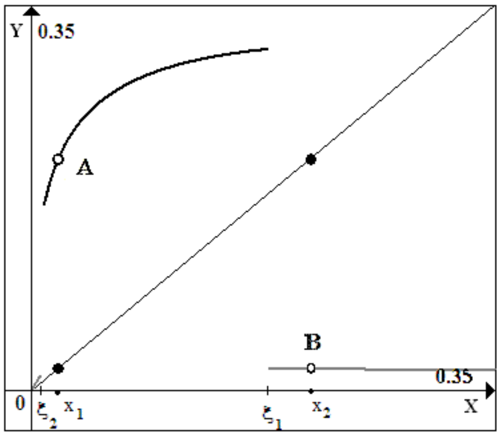

Figure 5: Different Profits for the Firms.
This is the set of parameters used: $k_1 = 0.024$, $k_2 = 0.562$, $r = 0.78$, $w = 0.15$. A stable 2-cycle on Δ and two stable symmetric fixed points *singly-generated* outside Δ, A and B.

Proposition 5 states that $TP^{(2)}(x_1, x_1) = TP(A) + TP(B)$. We also know that $TP(A) = TP(B)$ because they are symmetric, so that $TP^{(2)}(x_1, x_1) = 2TP(A) = TP^{(2)}(A) = 2TP(B) = TP^{(2)}(B)$. This means that the total profit after two periods is always the same, in each of the three cycles of the system, and thus for the market. This, however, does not mean that this equal total profit is distributed in the same way between the firms. If the system converges to the 2-cycle on the diagonal Δ, as we have already remarked, the two firms obtain the same profit each period (made up of the values $P_x(x_1, x_1) = P_y(x_1, x_1) = 0.1117$ and $P_x(x_2, x_2) = P_y(x_2, x_2) = 0.496$). As already remarked, in the 2-cycle we have $P_x^{(n)} = P_y^{(n)}$ for each $n > 0$. In the fixed point A the firm which produces more than the other obtains an higher profit as well: $P_y(A) = 0.5242463$ versus $P_x(A) = 0.0835145$. In B we have the same situation but in favour of the other firm, because of the symmetry. So, $P_y^{(2)}(A) = 1.04849$, whereas $P_x^{(2)}(A) = 0.167$, which clearly give $TP^{(2)}(A) = 1.215519$, showing that in the fixed point A the second firm takes 86% of total profit. Similarly, mutatis mutandis, in the fixed point B this behavior holds.

Summarizing, after two periods, total profit in the 2-cycle is equidistributed ($P_x^{(2)}(x_1, x_1) = P_y^{(2)}(x_1, x_1) = \frac{1}{2}TP^{(2)}(x_1, x_1) = 0.6077$), whereas in the fixed points one of the duopolists takes 86% of total profit. Let us remark that this occurs in a duopoly where the two firms are of different "size" in terms of the operated plants. In each fixed point one firm uses a low capacity limit and the other a higher one. In the next subsection we shall see a different case, in which the two fixed points belong to the diagonal, which means that either both choose a low capacity limit or both use a higher one.

5.2 Different attractors for the market

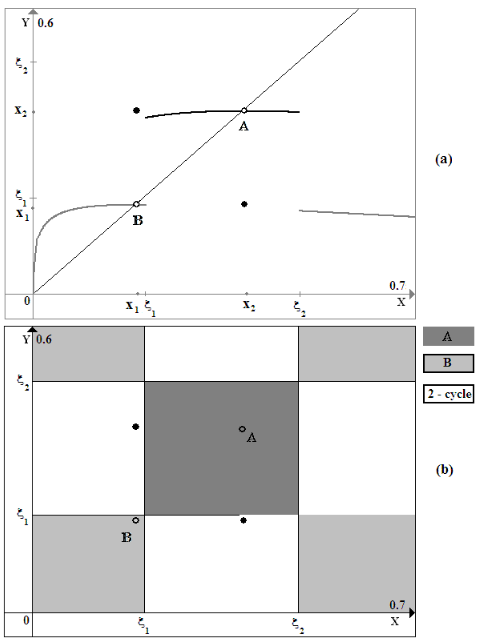

Figure 6: Different Profits for the Market (1).
Three coexisting attractors of the map T with parameters: $k_1 = 0.322$, $k_2 = 0.958$, $r = 0.1$, $w = 0.236$: two stable fixed points A and B on Δ, and a symmetric $2-$cycle *doubly-generated* outside Δ. In (b): The basins of attraction of the $2-$cycle, and the fixed points A and B, are shown in white, dark gray and light gray, respectively.

The situation shown in Fig. 6a, obtained with the set of parameters $k_1 = 0.322$, $k_2 = 0.958$, $r = 0.1$, $w = 0.236$, at a first sight looks quite similar to the previous case (shown in Fig. 5), while it is not. Now $\phi(x)$ has two asymptotically stable fixed points, so that (by Proposition 4) the two fixed points of T are on the diagonal, $A = (x_1, x_1)$ and $B = (x_2, x_2)$, and a stable 2-cycle *doubly-generated,* outside Δ and symmetric, exists: $\{(x_1, x_2), (x_2, x_1)\}$.

In each of the fixed points we have that $P_x^{(n)} = P_y^{(n)}$ for any $n > 0$ because they are located on Δ. The points of the 2-cycle are symmetric with respect to Δ so we also know that every 2 periods in the cycle the total profit is equidistributed: $P_x^{(2n)}(x_1, x_2) = P_y^{(2n)}(x_1, x_2) = \frac{1}{2}TP^{(2n)}(x_1, x_2)$. Thus we are not analyzing a situation of *different profits for the firms.*

We also have $TP(A) \neq TP(B)$, more specifically $TP(A) = 0.7252 > TP(B) = 0.508$, with $TP^{(2)}(A) = 1.4504$, $TP^{(2)}(B) = 1.016$. The total profits of the points of the 2-cycle are the same (because they are symmetric) and both equal to 0.6153, so that $TP^{(2)}(x_1, x_2) = 1.2306$. This means that even if the total profit is equidistributed in all the coexistent attractors, the total profit changes on the different cycles, and every two periods it is highest in the fixed point A. This is what we have called a situation of *different profits for the market.*

Probably this happens whenever $\phi(x)$ has more than one cycle, and especially coexisting attractors (because there is no reason to suppose that the coexisting cycles have the same total profit).

A second example in which we stress this difference of behaviors is related with the set of parameters $k_1 = 0.039$, $k_2 = 0.3$, $r = 0.923$, $w = 0.04222$. Now the one-dimensional map $x' = \phi(x)$ has two coexisting 2-cycles, both asymptotically stable, which correspond to two coexisting attracting 2-cycles on the diagonal for the map T, whose coordinates are $A = \{(x_1, x_1), (x_2, x_2)\}$ and $B = \{(y_1, y_1), (y_2, y_2)\}$ (see Fig. 7), and we expect two different total profits in these 2 cycles. In fact, after two periods we have $TP^{(2)}(y_1, y_1) > TP^{(2)}(x_1, x_1)$. Then the Cartesian product is formed by other twelve points, all belonging to attracting cycles of T of periods 1 or 2. In fact, by Propositions 3 and 4 we have two fixed points *singly-generated* by A, $A_1 = (x_1, x_2)$ and $A'_1 = (x_2, x_1)$, whose total profit after 2 periods is the same as that of the generating cycle A, i.e. $TP^{(2)}(A_1) = TP^{(2)}(A'_1) = TP^{(2)}(x_1, x_1)$, and similarly we have two fixed points *singly-generated* by B, $B_1 = (y_1, y_2)$ and $B'_1 = (y_2, y_1)$, whose total profit after 2 periods satisfies $TP^{(2)}(B_1) = TP^{(2)}(B'_1) = TP^{(2)}(y_1, y_1)$. Finally we have four distinct *doubly-generated* 2-cycles*:* $C_{A,B} = \{(x_1, y_1), (y_1, x_1)\}$ and $C_{B,A} = \{(x_2, y_2), (y_2, x_2)\}$ which are both symmetric, and, in addition, the two symmetric ones $C_{1,2} = \{(x_1, y_2), (y_1, x_2)\}$ and $C_{2,1} = C'_{1,2} = \{(x_2, y_1), (y_2, x_1)\}$, also reported in Fig. 7. In all these four different $2-$cycles total profit lies between total profits of the cycles on Δ (in particular, lower than the maximum one). In Table 1 we summarize the total profits of the market, on the 10 coexisting attractors of T, after two periods:

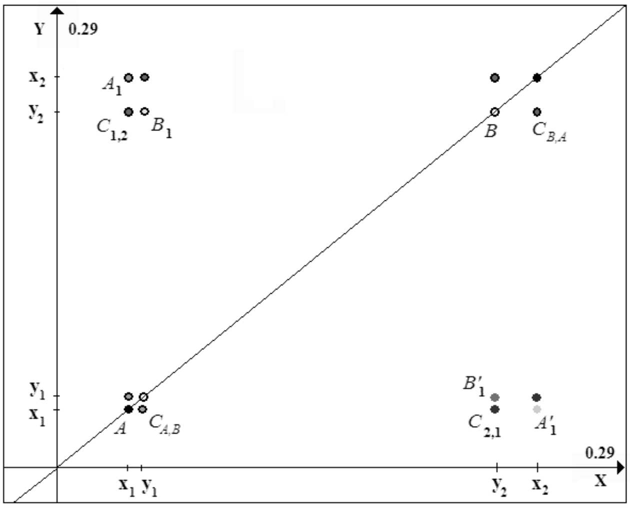

Figure 7: Different Profits for the Market (2).
10 coexisting attractors of the map T with parameters: $k_1 = 0.039$, $k_2 = 0.3$, $r = 0.923$, $w = 0.04222$.

Cycle	Period	Total profit after two periods
A	2	1.392
A_1	1	1.392
A'_1	1	1.392
B	2	1.466
B_1	1	1.466
B'_1	1	1.466
$C_{A,B}$	2	1.4291
$C_{B,A}$	2	1.4291
$C_{1,2}$	2	1.4292
$C_{2,1}$	2	1.4292

Table 1: Total profits

We can see (as in the first example of this subsection) that after two periods the *singly-generated* cyles give the same total profit of the "generating cycles", whereas from the *doubly-generated* cycles after two periods we get, from Table 1, a total profit which is between the total profits of each the two generators, leading to a situation of *different profits for the market.*

We remark that in general the cycles (attractors) in which the total profit is highest or lowest may also be external to the diagonal, and of *doubly-generated* types (even if our two examples do not suggest this).

6 Global Analysis

In all the examples considered up to now, we have seen the phenomenon of multistability. Indeed the shape of $\phi(x)$ far from the origin is "quite flat" and the periodic points of the one-dimensional function $\phi(x)$ are all in branches of the function such that $|\phi'(x)| < 1$.

From Proposition 5 it follows that all the coexisting cycles of T are asymptotically stable. In particular T cannot have saddle cycles (nor repelling nodes, and we recall that complex eigenvalues cannot occur in the class of maps having separate second iterate). This is the main difference in comparison to the class of maps in the continuous case. This means: *All the existing cycles are asymptotically stable and their basins fill in the phase-plane.*

A natural question then arises: Having no unstable cycles (the stable set of which usually give the frontier of different basins), what are the frontiers? As we shall see, the answer to this question, related with the global dynamic behavior, comes from the discontinuity points.

As it is well known, in presence of multistability, the attractor to which the system converges depends on the initial conditions. A small change in the initial conditions, or in the value of an exogenous parameter of the system, may drastically change the final situation for the duopolists. The first problem is related to the "basins of attractions" of the cycles in the phase plane, whereas the second is related to the bifurcations of the cycles which, in our model, are mainly (probably only) "border-collision bifurcations". In the first subsection we consider the first problem, showing how the frontiers of the basins are formed, and in the second subsection we shall briefly introduce the second problem, showing how the periodic points change, via border-collision bifurcations.

6.1 Basins of attraction

In order to detect the basins, the asymptotic behavior of the points of the phase plane of T can be better analyzed using the second iterate (i.e. the map T^2).

As remarked above, in our model it happens that all the existing cycles are asymptotically stable, it follows that the critical points of the function $\phi(x)$, or $\phi^2(x)$, do not have dynamic relevance, because such critical points converge to some stable cycle. Of primary importance are the discontinuity points of the function $\phi^2(x)$, which depend on those of $\phi(x)$. In fact the discontinuity points of $\phi^2(x)$ consist in all the *discontinuity points of* $\phi(x)$ *and in all their rank-1 preimages (when existing).* We recall that a discontinuity point, say $x = \xi$, of the reaction function $\phi(x)$ corresponds to the discontinuity line of equation $x = \xi$ in the phase plane for the two-dimensional map T and, due to the symmetry, also $y = \xi$ is a discontinuity line.

In general, at each discontinuity of the reaction function $\phi(x)$, say at $x = \xi$, we have to consider the two values associated with the "jump" of the function, say $\phi_l(\xi)$ (lower value) and $\phi_u(\xi)$ (upper value), and in the study of discontinuous maps the discontinuity points and their lower/upper values take the role of the "critical points" in the Julia sense, as considered in Mira et al 1996. If the asymptotic behavior of the two values $\phi_l(\xi)$ and $\phi_u(\xi)$ is the same, then the vertical line $x = \xi$ (discontinuity for T) plays no role in the basin frontiers. *Viceversa, if the asymptotic behavior of the two values* $\phi_l(\xi)$

and $\phi_u(\xi)$ are different, then the vertical line $x = \xi$ is a frontier of basins. This is clearly due to the fact that all the points in a right/left neighborhood of the discontinuity at $x = \xi$ have the same asymptotic behavior of the two values $\phi_u(\xi)$ and $\phi_l(\xi)$, or $\phi_l(\xi)$ and $\phi_u(\xi)$ (depending on the graph of the function). Such behaviors are to be considered in the function $\phi^2(x)$, for which $x = \xi$ is still a discontinuity point, and thus $\phi_l^2(\xi)$ and $\phi_u^2(\xi)$ the values at the jump.

Moreover if $x = \xi$ is a discontinuity point of $\phi(x)$ which is a separator of different basins, then also *any one of its rank-1 preimages is a separators of basins*, because points in a suitable right/left neighborhood of a preimage of $x = \xi$, say at $x = \xi_1^{-1}$, are mapped in one iteration in a right/left (or left/right) neighborhood of $x = \xi$, and thus have a different asymptotic behavior. The same reasoning can clearly be applied to the discontinuity points of the function $\phi^2(x)$. If one of the discontinuity points of $\phi^2(x)$ is a frontier of basins, *then the same property holds for all the existing preimages, of any rank.*

The asymptotic behavior of x, determined by the function $\phi^2(x)$ i.e. the segments of basins for this one-dimensional function, may be associated to the $x-$axis in our two-dimensional phase plane. In the symmetric case, for the map T, we also have that the asymptotic behavior of y is determined by the same function, so that the same segments of basins may be associated to the $y-$axis. Then we form the Cartesian product, so that we get the basins of the map T^2, from which we easily obtain those of T.

Regarding the points of the frontier itself, we confine the discussion to the discontinuity points: In each discontinuity point $x = \xi$ we have considered the two values $\phi_l(\xi)$ and $\phi_u(\xi)$ without saying which one is considered to be assumed in the discontinuity point ξ. Thus a point in a vertical frontier $x = \xi$ may behave as a point in its right/left neighborhood, depending on which value is taken at $x = \xi$. Similarly, in a horizontal frontier say $y = \xi$, we have that any point of a frontier may belong to any one of the basins of which it is a frontier, depending on the values associated at the discontinuity.

This generic description may be better understood by looking at the basins of some of the examples already seen in the previous sections. Let us return to the case represented in Fig. 5, where we have three coexisting attractors of T (a 2-cycle on the diagonal, and two symmetric fixed points outside). In Fig. 8a we show the graph of the function $\phi^2(x)$ with the discontinuity points at $x = \xi_1$ and $x = \xi_2$ as in the graph of $\phi(x)$ (see Fig. 5). Two more discontinuity point of $\phi^2(x)$ exist, at $x = \xi_1^{-1}$ and $x = \xi_2^{-1}$, rank-1 preimage of the discontinuity points $x = \xi_1$ and $x = \xi_2$, respectively, even if this last one is too close to zero to be seen in the graph. However, in the enlargement we show qualitatively that a sequence of preimages at $x = \xi_1^{-3k}$ and at $x = \xi_2^{-3k}$, for $k = 1, 2, ...$ must exist, rapidly approaching the origin.

From the graph of $\phi^2(x)$ in Fig. 8a we can see that the two values $\phi_l^2(\xi_1)$ and $\phi_u^2(\xi_1)$ (extrema at the jump in ξ_1) have a different asymptotic behavior: The lower value tends to x_1, while the upper value converges to x_2. Thus the

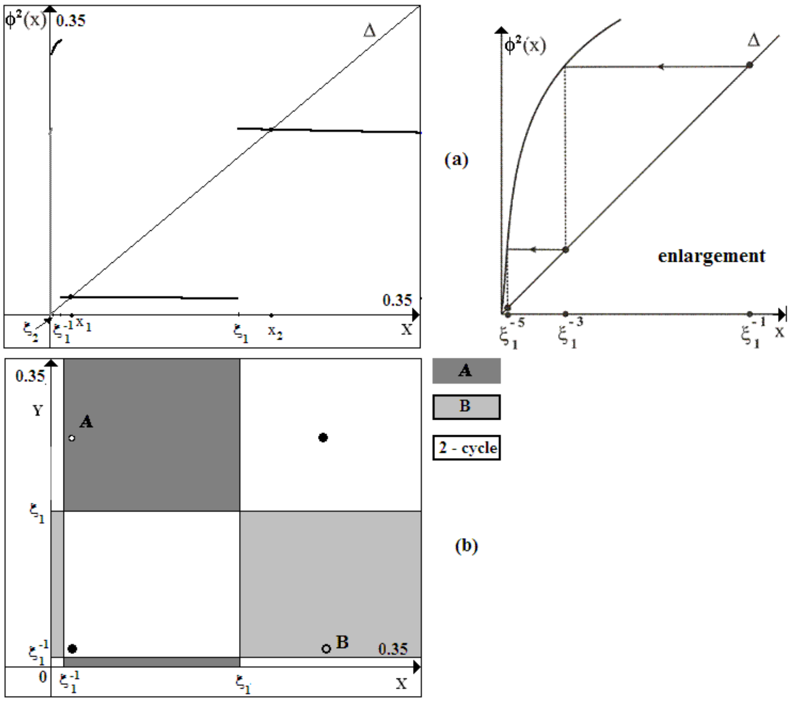

Figure 8: Basins' boundaries.
In (a) the graph of the function $\phi^2(x)$ is shown. In (b): The basins of attraction of the 2−cycle, and the fixed points A and B, are shown in white, dark gray and light gray, respectively.

line $x = \xi_1$ is a frontier of basins. The same occurs at the discontinuity in its rank-1 preimage, at $x = \xi_1^{-1}$. The frontiers of basins (though not discontinuity points) are all preimages $x = \xi_1^{-3k}$ for any $k > 1$ (even if these points are not visible in Fig. 8 because they are too close to zero). The discontinuity at $x = \xi_2$ plays no role: the upper and lower values at $x = \xi_2$ converge both to the same point x_2, and thus the same property holds at $x = \xi_2^{-1}$, and thus all the other preimages ξ_2^{-3k} are not involved in the formation of frontiers.

Summarizing; on the x-axis we have intervals separated by ξ_1, ξ_1^{-1}, ξ_1^{-3k}, the points of which tend alternatively (for $\phi^2(x)$) to $x = x_2$ and $x = x_1$. The sameholds about the $y-$axis, with the same intervals and the same asymptotic behaviors, so that the horizontal lines $y = \xi_1$, $y = \xi_1^{-1}$, $y = \xi_1^{-3k}$ approaching zero, belong to frontiers of basins. The Cartesian product is made up by rectangles, in which we can easily return to the basin for the map T (for which the points on the diagonal belong to the same 2-cycle), as shown in Fig. 8b (though the rectangles close to the coordinate axes are not visible).

Differing from this example, the one shown in Fig. 6a has a simpler structure of the basins, as shown in Fig. 6b. In fact, in this example the reaction function $\phi(x)$ has two discontinuity points at $x = \xi_1$ and $x = \xi_2$, and none has any rank-1 preimages (thus also the function $\phi^2(x)$ only has these two discontinuities). Moreover, it is immediately visible from Fig. 6a that the upper and lower values at both discontinuity points have different asymptotic behavior, so that both the lines $x = \xi_1$ and $x = \xi_2$ are frontiers, and no other vertical line exists. The same holds for the $y-$axis: the lines $y = \xi_1$ and $y = \xi_2$ are the only horizontal frontiers. Thus the rectangles shown in Fig. 6b give all the basins of the three coexisting cycles of T; the white region is the basin of the 2-cycle, the dark gray region the basin of the fixed point A and the light gray region the basin of the fixed point B.

6.2 Border-collision bifurcations

Up to now we have only shown single examples, at fixed parameter values, showing that all the periodic points of T always belong to asymptotically stable cycles. This is a very strong property of our model, due to the fact that the shape of the reaction function $\phi(x)$ in the periodic points is always very flat. This is however very difficult to prove in general, but we can support it by numerical evidence. Also in the case of only one fixed point of the duopoly (which, in the symmetric case, is necessarily a point on the diagonal) we know that in a wide range of significant values of the parameters, the fixed point is globally attracting. But as the numbers of involved plants change, i.e. as the parameters of the model are let to vary, we may have bifurcations, not due to a change in the value of the eigenvalue (i.e. of ϕ'), but due to the merging of the fixed point with a discontinuity (say at $x = \xi$, from which the term "border collision bifurcation") after which the fixed point disappears and something else happens to the trajectories, some other cycle appears, or more cycles appear, all stable (because of the flatness of the shape of the branches of the reaction function $\phi(x)$).

Similarly, as the parameters are further changed, a stable cycle never becomes "unstable" due to its eigenvalue, while it may undergo a "border collision bifurcation". This occurs when a periodic point of the cycle merges with a discontinuity point, after which the cycle no longer exists, and some other stable cycles appear. And so on. We cannot "predict" what occurs at a border collision bifurcation of a cycle; in general this depends on the kind of discontinuity, from a low value to a higher one or viceversa (from an high value to a lower one). It depends on the size "jump" at the discontinuity, i.e. on the magnitude of $(\phi_u(\xi) - \phi_l(\xi))$ at a discontinuity $x = \xi$.

This kind of bifurcations in discontinuous maps is a new research area, we refer to in Avrutin and Schanz, 2006, and in Avrutin et al. 2006, for some works related to a one-dimensional discontinuous normal form. Here we do not enter in deeper detail inside this problem; in this applied context we have simply verified numerically that several bifurcations of this kind occur in the cycles of the discontinuous one dimensional reaction function $\phi(x)$ (from which all the existing cycles of T can be obtained). In our model we have mainly four parameters: r, w, and the values of k_i. Fixing, for several different pairs, the values of the k_i, we have investigated the cycles existing on the diagonal as the other two parameters vary in a suitable interesting interval.

This leads to a two-dimensional bifurcation diagram (or two-dimensional orbit diagram, as it is often called, see Zhusubaliyev et al. 2007), an example of which is shown in Fig. 9 (the initial condition is a point on the diagonal and the numbers inside areas represent different periodicities of cycles). In this figure the capacity limits are fixed at the values $k_1 = 0.039$ and $k_2 = 0.3$, while $w \in [0, 1.25]$ and $r \in [0, 3]$. (For enlargement we only show $r \in [0, 2.65]$). Similar figures are also obtained for different values of k_i. In Fig.9 a few arrows can be seen: They indicate that thin regions of periodicity are not visible at this scale. (They look as boundaries, but are regions with cycles of different periods). In Fig. 9 different labels are also associated with fixed points

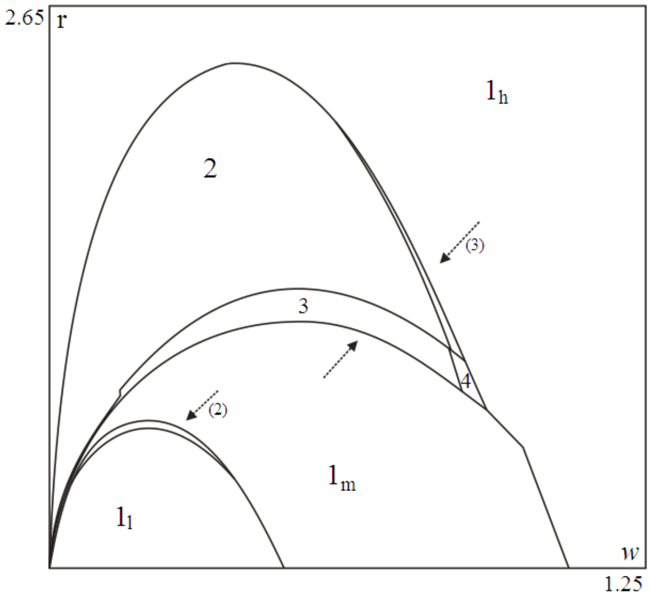

Figure 9: Two-dimensional orbit diagram in the (w, r) parameter plane.
We used $k_1 = 0.039$, $k_2 = 0.3$. Different labels denote an attracting cycle of $\phi(x)$ of different periodicity, or belonging to different branches of the reaction functions (1_h, 1_m and 1_l denote the fixed point corresponding to the higher, middle and lower capacity limit plant, respectively). The frontiers of the periodicity regions are curves of border-collision bifurcations. A few arrows indicate that thin regions of periodicity are not visible at this scale.

according to the plant selection used in the final attractor (according to which of the branches of $\phi(x)$ includes the periodic points). In particular, we can see three wide regions representing a fixed point, but belonging to three different branches of the reaction function. It is worth noting that the orbit-diagram is obtained using only one initial condition for $\phi(x)$ (or equivalently on the diagonal for T). Thus multistabilty in $\phi(x)$ is not represented here, however we know that it occurs (we have shown above examples with two coexisting stable fixed points and two coexisting stable two-cycles for $\phi(x)$). The global bifurcations occurring on the borders separating the different periodicity areas represented in Fig. 9 are all border-collision bifurcations, due to the existence of discontinuity points of the map $\phi(x)$, and thus of T.

We leave this kind of analysis for future research. Here we prefer to show that the dynamic properties analyzed are not peculiar for the symmetric case alone, as they also occur in the generic case, with firms having plants with different capacity limits, as briefly illustrated in the next session.

7 The generic case

In this section we extend the analysis to the generic case in which the duopolists do not have identical plants, which is a more realistic situation, represented by the map M in (19). To do this we start from a symmetric case, and then change the capacity limits for the firms. This "symmetry breaking" has no particular effect: no "drastic changes" occur. This is expected, because the diagonal in phase space is no longer invariant, the only effect is that the attractors on the diagonal move outside, and the symmetry with respect to Δ is broken. However the map M is still a map with the properties of separate second iterate and, as already remarked, many of the properties proved in Bischi et al. 2000 still hold, although the maps F and G given in (20) are piecewise smooth *discontinuous functions*. In particular, symmetry is transformed into the property that if I is an invariant set for $F(x)$ then $\phi_2(I)$ is invariant for $G(y)$, so it is enough to study one of the one-dimensional maps, say F, to get the properties of the two-dimensional map M. Clearly now a discontinuity point, say $x = \xi$, of the reaction function $F(x)$ corresponds to the discontinuity line $x = \xi$ for the two-dimensional map M in the phase plane (x, y), and then $y = \phi_2(\xi)$ is a discontinuity point of $G(y)$ to which corresponds the discontinuity line of equation $y = \phi_2(\xi)$ in the phase plane, for M.

Another immediate result, because the proof makes no use of the continuity of the functions, is the number and structure of the cycles of M given the cycles of F. Assuming that the cycles do not have periodic points in the discontinuity (i.e. the cycle is not at a bifurcation), then also the local stability analysis is the same, so that multistability is still a peculiarity. But it is clear that (as in the symmetric case, seen in the previous sections) the bifurcations occurring to the cycles of F (and thus of M) are very different, because they are associated with the merging of periodic points into the discontinuity points, so that they are "border-collision bifurcations" occurring in the one-dimensional map F (or in the discontinuity lines of M). This is expected, as $\phi(x)$ in the symmetric case, now also the map $F(x)$ has a "flat" shape, so that we observe *only stable cycles*. Thus, multistability and the existence of only stable cycles show that also the properties of the basin frontiers are still the same. This means that the frontiers are associated with the discontinuity points of F (and thus G) and to the related preimages (when they exist).

Let us show an example, starting from the situation represented in Fig. 5. If the capacity limits of a duopolist are such that $0.015 < k_{x,1} < 0.131$ or $0.41 < k_{x,2} < 1.97$, keeping fixed at the values of the other: $k_{y,1} = 0.024$ and

$k_{y,2} = 0.562$, $r = 0.78$, $w = 0.15$, we still have a two-cycle which coexists with two fixed points (see Fig. 10). The 2−cycle is no longer on the diagonal Δ, and the fixed points A and B external to Δ are no longer symmetric. The fixed points are shown at the intersection of the two reaction functions in Fig. 10. The coordinates of the cycles come from the periodic points of $F(x)$, whose graph is shown in Fig. 11a. The 2−cycle of M is thus $\{(x_1, y_1), (x_2, y_2)\}$ where x_1 and x_2 are the coordinates of the fixed points of $F(x)$ (see Fig. 11a), $y_1 = \phi_2(x_1)$ and $y_2 = \phi_2(x_2)$, whereas $A = (x_1, y_2)$ and $B = (x_2, y_1)$ are the two fixed points of M. Regarding to the basins of the three different

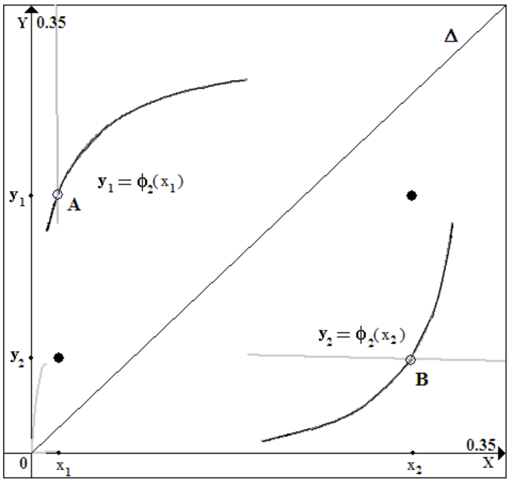

Figure 10: Coexistence in the generic case.
Three coexisting attractors of the map M (generic case) with parameters: $k_{x,1} = 0.114$, $k_{x,2} = 0.562$, $k_{y,1} = 0.024$, $k_{y,2} = 0.562$, $r = 0.78$, $w = 0.15$: A stable 2−cycle and two stable fixed points A and B.

attractors of M, we can reason exactly as in the symmetric case. Considering the map M^2 we can refer to the graph of $F(x)$ (see Fig. 11a), where we can see that two discontinuity points exist, at $x = \xi_1$ and $x = \xi_2$. In both discontinuity points, the lower and upper values at the jump have different asymptotic behavior; thus the vertical lines $x = \xi_1$ and $x = \xi_2$ in the phase plane are frontiers of basins, and the same occurs for the horizontal lines $y = y_1$ ($y_1 = \phi_2(x_1)$) and $y = y_2$ ($y_2 = \phi_2(x_2)$). Moreover, these two points of discontinuity ξ_1 and ξ_2 have infinitely many preimages of any rank, ξ_1^{-n} belonging to $F^{-n}(\xi_1)$ and ξ_2^{-n} belonging to $F^{-n}(\xi_2)$ for $n = 1, 2, ...$ which approach the origin: Thus, although not visible in the basins shown in Fig. 11b (because they are too close to the coordinate axes), also the vertical lines $x = \xi_1^{-n}$ and $x = \xi_2^{-n}$ in the phase plane are frontiers of basins, and the same occurs for the horizontal lines $y = \phi_2(\xi_1^{-n})$ and $y = \phi_2(\xi_2^{-n})$.

We remark that now, the more the production opportunities of the firms differ from the symmetric case, the more the relative profits of the two firms can comparatively change, as compared to the case presented in the previous sections. We also recall that for the map M the existence of only one stable

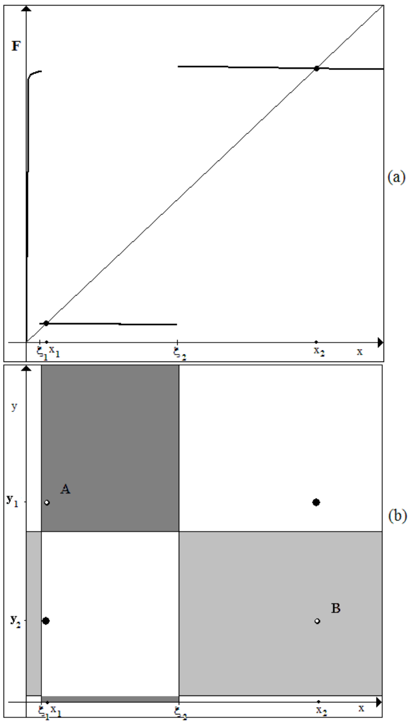

Figure 11: Basins' boundaries in the generic case.
In (a): The graph of the one-dimensional map F governing the dynamics of M in the case shown in Fig.10. In (b): The basins of attraction of the $2-$cycle, and the fixed points A and B, are shown in white, dark gray and light gray, respectively.

fixed point, not on the diagonal is a possibility. We can see this as the effect of a border-collision bifurcation occurring when we change the value of $k_{x,1}$ or $k_{x,2}$ further, exiting from the ranges given above. In fact, if we decrease the value of $k_{x,1}$ below 0.015, in the graph of $F(x)$ the values of the fixed points x_1 and x_2 decrease, approaching the discontinuity points ξ_1 and ξ_2, so that the points of the $2-$cycle of M as well as the fixed points, approach the discontinuity lines of the map (which are $x = \xi_1$, $x = \xi_2$ $y = \phi_2(\xi_1)$ and $y = \phi_2(\xi_2)$). Thus a border-collision bifurcation occurs, whose effect, when continuing to decrease the parameter $k_{x,1}$, is to leave a fixed point in the plane A of M as the unique, globally stable attractor, with coordinates $(x^*, \phi_2(x^*))$. It is the unique intersection point of the reaction functions and is located in the upper left side of the phase plane (close to the old point A).The effects are similar if the value of $k_{x,1}$ is increased above 0.131 : In the graph of $F(x)$ the value of the fixed points of $F(x)$ approach the discontinuity points, so that all the cycles of M undergo a border-collision bifurcation, whose effect, continuing to increase the parameter $k_{x,1}$, is to leave a fixed point B of M as unique attractor, globally stable in the plane. Its coordinates are $(x^*, \phi_2(x^*))$; it is unique intersection point of the reaction functions, and is located in the lower right side of the phase plane (close to the old point B). Also keeping

$k_{x,1}$ fixed and increasing or decreasing $k_{x,2}$, exiting from the given ranges, we get the same kind of border collision bifurcations, with the same effects as those described above (when $k_{x,1}$ is varied).

It is worth noting that in the symmetric case, when the two firms both have identical capacity limits, a globally attracting fixed point necessarily was located on the diagonal, i.e, both firms used identical plant operations, and both firms had the same private profit. In this generic case, a globally attracting fixed point is associated with firms having different capacity limits and they differ in terms of private profits. The above example shows that the position of the unique globally attracting fixed point is different depending on the variation in the parameters.

8 Conclusions

In this work we have considered a Cournot duopoly under an isoelastic demand function and cost functions with built-in capacity limits, which can be derived from CES production functions. The special feature is that each firm is assumed to operate multiple plants. Each firm has two plants with different capacity limits, which may be used one by one (denoted numbers 1 and 2), or in combination (number 3). It has three cost options, the third being to run both plants, dividing the production load according to the principle of equal marginal costs. As a consequence the marginal cost functions come in three disjoint pieces, so the reaction functions, derived on basis of global profit maximization, can consist of several disjoint pieces. Thus we are faced with discontinuous, piecewise smooth, reaction functions.

We first analyzed in detail the case of identical firms, characterized by a symmetric two-dimensional map, under which several stable cycles may coexist, and gave explicit formulations of how to detect all the cycles starting from the reaction function. From any point of the phase space a trajectory converges to some stable situation. Then it becomes fundamental to compare the coexistent periodic attractors in terms of profits, and we have shown that a unique "best situation" may not always be found.

Also it becomes fundamental to know the basin structure, in order to know what each initial state will converge to. A property of this map is the total absence of unstable cycles We have shown that the frontiers of the different basins (an example with 10 coexistent attractors was also given) are related with the discontinuities of the reaction functions, explaining how to detect the frontiers. Another feature of the duopoly model is that the changes in the coexistent attractors never occur due to some stable cycle becoming unstable, instead, all the bifurcations are due to border collision; merging of some cycle with the discontinuity points. This is still an open research field, and more analysis may be performed for this model. Briefly we have also shown how all these properties persist in the generic case, both in terms of coexistence of stable cycles, the structure of the basin boundaries, and border collision bifurcations. We are conscious that also this generic case may be better investigated, and we leave this as future research work.

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