UNIVERSITÀ DEGLI STUDI DI ANCONA

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INVESTMENT AND UNCERTAINTY WITH RECURSIVE PREFERENCES

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Abstract

This paper analyses the relationship between uncertainty and investment when firms are risk averse and have a constant return to scale technology. Using recursive preferences, the paper demonstrates that not only the degree of risk aversion is important in determining the sign of the investment-uncertainty relationship but that the intertemporal substitution elasticity also plays a crucial role. The model presented suggests the existence of a positive relationship between investment and uncertainty for reasonable values of the parameters. In the second part of the paper we extend the analysis taking into consideration the presence of adjustment costs in the investment process. Except for particular values of the preference parameters, the uncertainty-investment relationship has a not a definite sign anymore.

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Investment and uncertainty with recursive preferences

Davide Ticchi

1 Introduction

Economic theory investigated the investment-uncertainty relationship for almost forty years. However, there is not a definite result until now. Economic literature on this field began with the work of Oi (1961) and the following papers of Hartman (1972) and Abel (1983): they showed that an increase in output price uncertainty raises the investment of a risk-neutral competitive firm with constant returns to scale technology. This is due to the convexity of the marginal product of capital with respect to the output price (or, in general, the stochastic variable). By Jensen's inequality, a greater price variability implies an higher expected value of the marginal product of capital, and so an higher investment. This result follows from the assumption that the labor is a perfect flexible input: indeed, if the firm can substitute capital with labor, it will adjust labor after observing the price shock, and the marginal product of capital increases more than proportionally relative to price.

This theoretical conclusion has been contadicted by empirical research: most papers show the existence of a negative relationship between investment and uncertainty (see, for example, Pindyck and Solimano (1993), Ferderer (1993b), Leahy and Whited (1996), Guiso and Parigi (1999)). Economic theory identified two factors that potentially could reverse the positive effect of uncertainty on investment: irreversibility and risk aversion. In the first case, it has been shown (see Bernanke (1983), McDonald and Siegel (1986), Bertola (1998), Pindyck (1988)) that, if the firm cannot resell its capital goods, then investing now implies "killing" the option of investing in the future. Clearly, this increases the (opportunity) cost of investment. Since the value of the option increases with price volatility, greater uncertainty will increase the cost of investment and reduce its level. Risk aversion could depress investment too (see Craine (1989), Zeira (1990), Nakamura (1999)): indeed, if the firm (or outside investors) is risk averse, then higher price variability implies more risk and less investment.

However, both these factors are not by themselves sufficient conditions to have uncertainty and investment negatively related. Two reasons explain this result. First, the introduction of irreversibility or risk aversion (with an opposite sign to that indicated by Hartman and Abel) does not necessarily reverse the effect of uncertainty on investment. But there is a second, more subtle reason. Simply considering investment decisions as irreversible and not taking into account their intertemporal nature may not be sufficient to give irreversibility its proper role. As noted by Pindyck (1993), the absence of "linkages between one period and the next [...] eliminates irreversibility from the problem".

In this paper we intend to demonstrate that something similar occurs with risk aversion. Indeed, the introduction of risk aversion into the dynamic set-up of investment decisions without distinguishing its role from that of intertemporal substitutability can give rise to situations where risk aversion produces effects opposite to those suggested by the traditional economic theory.[1] As we will see, in the traditional expected utility set-up with $CRRA$ preferences high values of risk aversion determine a positive relationship between uncertainty and investment. The use of recursive preferences that allow us to separate risk aversion and intertemporal substitutability gives the possibility to show that an high degree of risk aversion always reduces the expected utility of future profits when uncertainty increases. However, the effects of greater uncertainty on investment depend also on intertemporal substitution elasticity, and it is possible that investment reacts positively to uncertainty even if risk aversion is high. This mechanism cannot be captured by the traditional expected utility framework as risk aversion and intertemporal substitution elasticity collapse in one parameter.

To illustrate these points, we use a general equilibrium model where firms decide the optimal investment level and consumers (which are the owners of the firms) choose their consumption and portfolio allocation.

In the second part of the paper we extend the analysis taking into consideration the presence of adjustment costs in the investment process. The uncertainty-investment relationship has a not a definite sign anymore in this framework. However, under irreversibility and a unitary consumer's intertemporal substitution elasticity this relationship becomes negative.[2]

This paper is organized as follows. In Paragraph 2 we consider the problem for the consumer and the firm without investment adjustment costs. Then the effect of uncertainty on investment is analyzed. In Paragraph 3 the model is extended to the case where there are adjustment costs. Paragraph 4 analyses the effect of uncertainty on investment under irreversibility. Paragraph 5 concludes. The Appendixes contain some more technical derivations.

[1] The importance of risk aversion was already recognized by Oi (1961) using a static model.

[2] The irreversibility is introduced by assuming that the price of capital is zero and adjustment costs are quadratic as in Abel (1983). This implies that the firm never finds convenient to disinvest which implies that the lower bound for the investment is zero.

2 Reversible investment

We assume that in the economy it is produced only one good that can be used for consumption or investment. We also assume that capital does not depreciate and therefore the capital accumulation constraint is $dK = Idt$. All the firms in the economy have a Cobb-Douglas constant return to scale production function $\vartheta F(K, L) = \vartheta K^{1-\alpha} L^{\alpha}$, where K is the stock of capital, L is the amount of labor employed and ϑ is a multiplicative technological shock to the production function. Under these conditions the firms operating profit, i.e. revenues minus the cost of the variable factors of production, is $\pi = \vartheta^{\beta} K$, with $\beta = \frac{1}{1-\alpha}$.[3]

The technological shock ϑ follows a geometric Brownian stochastic motion

$$\frac{d\vartheta}{\vartheta} = \sigma dz \tag{1}$$

where dz is a standard Wiener process.

Using Ito's lemma we obtain the following stochastic process for ϑ^{β}

$$\frac{d\vartheta^{\beta}}{\vartheta^{\beta}} = \beta(\beta-1)\frac{\sigma^2}{2}dt + \beta\sigma dz \equiv mdt + sdz \tag{2}$$

where m is the istantaneous growth rate of output due to the technological progress and s is its variability.

The hypothesis that all the firms are equal allow us to consider only the behavior of the representative firm. This one chooses its optimal investment so as to maximize the present discounted value $V(\vartheta_t, K_t)$ of the expected cash flow:

$$V(\vartheta, K) = \max_{\{I\}} E_t \left\{ \int_t^{\infty} \exp\left[-\int_t^s \check{r}(u) \cdot du \right] \left(\vartheta^{\beta} K - I \right) \cdot ds \right\} \tag{3}$$

[3] The implicit hypothesis is that the firm observe the realization of the shock and consequently adjust the amount of labor. Under these conditions, the operating profit (i.e. revenues minus the cost of variable inputs) is obtained by solving the following maximization problem $\chi = \max_L \left\{ \vartheta K^{1-\alpha} L^{\alpha} - wL \right\}$, where w is the real wage. Then we get $\chi = (1-\alpha)\left(\frac{w}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \vartheta^{\frac{1}{1-\alpha}} K = \eta \vartheta^{\beta} K$. If we assume that the supply of labor is such that the real wage w is constant, then η is also constant and we can normalize it to one without loss of generality.

where $\check{r}$ is the rate of return required by the market to finance the firm.

The correspondent Bellman equation for this problem is

$$\check{r}V\,dt = \max_{\{I\}}\left\{\left(\vartheta^{\beta}K - I\right)dt + E_t\left(dV\right)\right\} \tag{4}$$

The left hand side is the return required by the market while the right hand side is the expected return of the firm. The latter is given by the sum of the firm's dividend $(\vartheta^{\beta}K - I)$ and the expected capital gain $(E_t(dV))$. The absence of arbitrage possibilities implies the equality between these two components.

The expected capital gain can be determined using Ito's lemma and taking into account the stochastic process (1) of the technological shock ϑ:

$$\begin{aligned} E_t\left(dV\right) &= E_t\left[V_K \cdot dK + V_{\vartheta}\cdot d\vartheta + \frac{1}{2}V_{\vartheta\vartheta}\cdot\left(d\vartheta\right)^2\right] = \\ &= V_K\cdot dK + \frac{1}{2}\sigma^2\vartheta^2 V_{\vartheta\vartheta}\cdot dt \end{aligned} \tag{5}$$

Substituting (5) into (4) and using $dK = Idt$ we get the equation

$$\check{r}V = \max_{\{I\}}\left\{\vartheta^{\beta}K - I + V_K I + \frac{1}{2}\sigma^2\vartheta^2 V_{\vartheta\vartheta}\right\} \tag{6}$$

Then, the maximization of (6) with respect to I leads to the following first order condition

$$V_K = 1 \tag{7}$$

This condition implies that the marginal value of the firm (i.e. the marginal value of the installed capital) is equal to the price of capital. In this sense condition (7) has a similar interpretation to the Tobin's marginal q.

Using (7) we can rewrite (6) as

$$\check{r}V = \vartheta^{\beta}K + \frac{1}{2}\sigma^2\vartheta^2 V_{\vartheta\vartheta} \tag{8}$$

Equation (8) is an ordinary second order differential equation in ϑ with variable coefficients. If we exclude the speculative bubbles, the only solution economically relevant is (see Appendix A for details)

$$V = \frac{\vartheta^{\beta}K}{\check{r} - m} \tag{9}$$

The fundamental solution gives an expected rate of return equal to $\check{r}$. The instantaneous rate of dividend is $\frac{1}{V}\left(\vartheta^{\beta}K - I\right)$ while the capital's expected instantaneous rate of return is $\frac{1}{dt}\frac{E_t(dV)}{V} = \frac{I}{V} + m$.[4] If we sum dividends and

[4]For further details see Appendix B.

expected capital gains we obtain

$$\frac{1}{V}\left(\vartheta^{\beta}K - I\right) + \frac{1}{dt}\frac{E_t\left(dV\right)}{V} = \frac{\vartheta^{\beta}K}{\vartheta^{\beta}K/\left(\check{r} - m\right)} + m = \check{r} \tag{10}$$

as implied by the Bellman equation (4).

Computing V_K from (9) and using the optimal condition $V_K = 1$ we get

$$\frac{\vartheta^{\beta}}{\check{r} - m} = 1 \tag{11}$$

and then

$$\check{r} = \vartheta^{\beta} + m \tag{12}$$

Equation (12) determines the optimal amount of capital K^*.

2.1 The consumers

We assume that at each point in time t the representative consumer maximizes the intertemporal objective $U(t)$ defined by the recursion[5]

$$f\left[\left(1 - R\right)U\left(t\right)\right] = \frac{1 - R}{1 - \frac{1}{\varepsilon}}C\left(t\right)^{1 - \frac{1}{\varepsilon}}h + e^{-\rho h}f\left[\left(1 - R\right)E_t U\left(t + h\right)\right] \tag{13}$$

where the function $f\left(x\right)$ is defined as

$$f\left(x\right) = \frac{1 - R}{1 - \frac{1}{\varepsilon}}x^{\frac{1 - \frac{1}{\varepsilon}}{1 - R}} \tag{14}$$

In (13), $C\left(t\right)$ is the consumption at time t, $\rho > 0$ is the subjective rate of time preference, $R > 0$ is the coefficient of relative risk aversion, $\varepsilon > 0$ is the intertemporal substitution elasticity, and h the length of the economic decision interval.[6]

Consumers can allocate their saving between a riskless asset which yields an instantaneous rate of return equal to r and a risky asset (i.e. the stocks of the firms) whose expected instantaneous rate of return is $\check{r}$.

If W is the consumer's wealth then her budget constraint is:

$$dW = \left[a\check{r}W + \left(1 - a\right)rW - C\right]\cdot dt + aWs\cdot dz \tag{15}$$

[5]The particular preferences used are due to Obstfeld (1994b). For a more general preference setup see Epstein and Zin (1989), Weil (1989) and Weil (1990).

[6]Clearly, E_t is a mathematical expectation conditional on time-t information.

where a is the share of wealth invested in the risky asset.[7] Equation (15) shows that saving, dW, is equal to the portfolio income minus consumption plus a zero expected value stochastic component generated by the existence of the risky asset in the portfolio.[8]

Let $J(W_t)$ denote the maximum feasible level of lifetime utility when the wealth at time t equals W_t.[9] Using the Ito's lemma in the maximization of $U(t)$ in (13) we obtain the Bellman equation

$$\begin{aligned}\rho \cdot f\left[(1-R)\, J(W)\right] &= \max_{a,C}\{\frac{1-R}{1-\frac{1}{\varepsilon}} C^{1-\frac{1}{\varepsilon}} + (1-R)\cdot f'\left[(1-R)\, J(W)\right]\cdot \\ &\cdot\{J'(W)\left[a\tilde{r}W + (1-a)\, rW - C\right] + \\ &\frac{1}{2} J''(W)\, a^2 s^2 W^2\}\} \qquad (16)\end{aligned}$$

From (16) we derive the following first order conditions with respect to a and C respectively:

$$J'(W)(\tilde{r}-r) + J''(W)\, as^2 W = 0 \qquad (17)$$

$$C^{-\frac{1}{\varepsilon}} - f'\left[(1-R)\, J(W)\right] J'(W) = 0 \qquad (18)$$

Equation (13)'s form suggests a guess that maximized lifetime utility U is given by:

$$J(W) = \frac{(bW)^{1-R}}{1-R} \qquad (19)$$

for some constant $b > 0$. The use of (19) in (17) allows us to get the demand for the risky asset

$$a = \frac{\tilde{r}-r}{Ru^2} \qquad (20)$$

which is a constant fraction of wealth. From equations (19) and (18) it is possible to derive the following consumption function

$$C = b^{1-\varepsilon} W \qquad (21)$$

Equation (21) reveals that the consumption is a constant fraction $\psi \equiv b^{1-\varepsilon}$ of wealth:

[7] For simplicity, we have omitted the labor income in the budget constraint. Although this might seem strange we need to clarify that under some mild assumptions on the utility function the results that would be obtained with the explicit introduction of labor are the same.

[8] For details about the budget constraint see Appendix B.

[9] In the following part of the paper the subscript t for the time will be omitted when it does not create confusion.

$$\psi = \frac{C}{W} \qquad (22)$$

Substitution of equations (19), (20) and (21) into equation (16) implies that

$$\psi = \varepsilon\rho - (\varepsilon - 1)\left[r + \frac{(\tilde{r}-r)^2}{2Rs^2}\right] \qquad (23)$$

and confirms that the value function is

$$J(W) = \frac{\left[\psi^{1-\frac{1}{\varepsilon}} W\right]^{1-R}}{1-R} \qquad (24)$$

Equations (20) and (21) highlight that consumption behavior depend both on intertemporal substitution elasticity and relative risk aversion coefficients while portfolio choices only by the risk aversion parameter.

From equation (20) it is possible to derive the instantaneous rate of return on the risky asset:

$$\tilde{r} = r + aRs^2 \qquad (25)$$

Equation (25) shows that the expected rate of return $\tilde{r}$ on the risky asset is equal to the riskless interest rate r plus a risk premium (aRs^2) positively related to the uncertainty and risk aversion degree.

Under the assumption that the net supply of the riskless asset is zero (and therefore that the consumer's endowment are given only by the stocks of the firms) the net demand of the riskless asset has also to be zero in equilibrium. Therefore, $1 - a = 0$ and $a = 1$ and from equation (25) it follows that

$$\tilde{r} = r + Rs^2 \qquad (26)$$

Moreover, the equilibrium in the assets market implies that the consumer's wealth is equal to the market value of the firms $V = W$ and then the consumption function is

$$C = \psi W = \left\{\varepsilon\rho - (\varepsilon - 1)\left[r + \frac{(\tilde{r}-r)^2}{2Rs^2}\right]\right\} V \qquad (27)$$

2.2 The effect of uncertainty on investment under reversibility

In the traditional neoclassical model the investment is determined by saving in the good market:

$$I = \vartheta^{\beta} K - C = \vartheta^{\beta} K - \psi V \tag{28}$$

From the problem of the firm we know that $V = \frac{\vartheta^{\beta} K}{\tilde{r} - m}$. As all firms are identical V also represents the aggregate value of the firms. In equilibrium $K = K^{*}$ and the optimal accumulation path equation (12) holds, i.e. $\tilde{r} = \vartheta^{\beta} + m$. Therefore, the value of the firm is

$$V = \frac{\vartheta^{\beta} K}{\tilde{r} - m} = \frac{\vartheta^{\beta} K}{\vartheta^{\beta}} = K \tag{29}$$

Substitution of equation (29) into (28) leads to

$$I = \left(\vartheta^{\beta} - \psi\right) K \tag{30}$$

If we plug equations (12) and (26) into equation (30) we obtain the following investment function

$$\begin{aligned} I &= \varepsilon \left\{ \vartheta^{\beta} - \rho + \left(1 - \frac{1}{\varepsilon}\right) m - \frac{1}{2}\left(1 - \frac{1}{\varepsilon}\right) R s^{2} \right\} K = \\ &= \varepsilon \left\{ \vartheta^{\beta} - \nu \right\} K \end{aligned} \tag{31}$$

where $\nu = \rho - \left(1 - \frac{1}{\varepsilon}\right) m + \frac{1}{2}\left(1 - \frac{1}{\varepsilon}\right) R s^{2}$ can be interpreted as a risk-adjusted discount rate or risk-adjusted cost of capital (see Appendix C). Equation (31) implies that the investment is positive when the marginal product of capital ϑ^{β} is greater than the user cost of capital ν, and vice-versa.

The consumption function (27) becomes

$$C = \psi K = \varepsilon \left\{ -\left(1 - \frac{1}{\varepsilon}\right) \vartheta^{\beta} + \nu \right\} K \tag{32}$$

The effect of an increase in uncertainty on investment is

$$\frac{dI}{d\sigma^{2}} = -\frac{dC}{d\sigma^{2}} = -\varepsilon \frac{d\nu}{d\sigma^{2}} K \tag{33}$$

The sign of the investment-uncertainty relationship depends on the effect of uncertainty on the risk-adjusted discount rate: if an increase in uncertainty leads to an higher risk-adjusted discount rate then the investment decreases, and vice-versa.

The effect of the uncertainty on the risk-adjusted discount ratê is given by the following relationship:

$$\frac{d\nu}{d\sigma^{2}} = \frac{1 - \varepsilon}{\varepsilon} \left(\frac{dm}{d\sigma^{2}} - \frac{1}{2} R \frac{ds^{2}}{d\sigma^{2}} \right) = \frac{1 - \varepsilon}{\varepsilon} \frac{\alpha - R}{2\left(1 - \alpha\right)^{2}} \tag{34}$$

Using the last two equations, we get

$$\frac{dI}{d\sigma^{2}} = -\left(1 - \varepsilon\right) \frac{\alpha - R}{2\left(1 - \alpha\right)^{2}} K \tag{35}$$

Making the substitution $R = \frac{1}{\varepsilon}$ allows us to see what would have been this result in the expected utility set-up with preferences of the $CRRA$ type, with R being the coefficient of relative risk aversion:[10]

$$\frac{dI}{d\sigma^{2}} = \frac{\left(1 - R\right)\left(\alpha - R\right)}{2R\left(1 - \alpha\right)^{2}} K \tag{36}$$

This is exactly the Nakamura (1999)'s result. Comparing this last equation with (35), we can see the difference between the two approaches: the effect of uncertainty on the risk-adjusted discount ratê, and therefore the sign of the investment-uncertainty relationship, depends not only from the comparison between the coefficient of relative risk aversion R and the elasticity of output to labor α, but also from the magnitude of the intertemporal substitution elasticity ε.

From equation (34) we know that when uncertainty increases two things happen: the instantaneous growth rate of output increases ($\frac{dm}{d\sigma^{2}} > 0$) and therefore the rate of return on investment also increases, but the higher uncertainty implies an higher cost for variability to the individual equal to $\frac{1}{2} R \frac{ds^{2}}{d\sigma^{2}}$. The difference $\frac{\alpha - R}{2(1-\alpha)^{2}}$ can be interpreted as the variation of the adjusted instantaneous growth rate of output ($AIGRO$).[11] Notice that the role of the risk aversion parameter R has an effect ouly on the amount by which this growth rate is adjnsted downward to reflect the negative effect of more volatility. The effects on investment of changes in the $AIGRO$ depend only on the magnitude of the intertemporal substitution elasticity parameter ε. Other things equal, if the $AIGRO$ is positive then lower ε implies that marginal utility is declining more quickly. This implies an higher level of current consumption and consequently a lower level of the investment. The effect of lower ε is reversed when $AIGRO$ is negative.

In particular, if $R < \alpha$ then when uncertainty increases the instantaneous growth rate of output increases more than the cost for the higher variability.

[10] If $R = \frac{1}{\varepsilon}$ then equation (13) implies that as $h \to 0$, $U(t)$ approaches to the expected utility form

$$U\left(t\right) = E_{t} \left\{ \frac{1}{1 - R} \int_{t}^{\infty} C\left(s\right)^{1-R} e^{-\delta(s-t)} ds \right\}.$$

[11] For an interesting similar application see Obstfeld (1994a).

$\frac{dI}{d\sigma^2}$	$R > \alpha$	$R = \alpha$	$R < \alpha$
$\varepsilon < 1$	> 0	$= 0$	< 0
$\varepsilon = 1$	$= 0$	$= 0$	$= 0$
$\varepsilon > 1$	< 0	$= 0$	> 0

Table 1:

This leads to two effects: *a*) a positive wealth effect which causes an increase in consumption and a reduction of current investment as, all else the same, the increase in uncertainty has increased his future income as the *AIGRO* is positive; *b*) a negative substitution effect due to the fact that the higher rate of return on investment makes more convenient for the individual to reduce the consumption and increase the investment. Which of the two effects prevail depends on the magnitude of the intertemporal substitution parameter. If the intertemporal substitution elasticity parameter ε is low (i.e. less than one) then the marginal utility is declining very quickly (as the *AIGRO* is positive) and the wealth effect is greater than the substitution effect. Therefore, the individual increases consumption and reduces investment. If ε is high enough (i.e. greater than one) then the cost of shifting consumption over time is relatively low and the substitution effect prevails on the wealth effect: in this case the effect on investment is positive. When $\varepsilon = 1$ the substitution and wealth effects exactly offset each other and no change in consumption and investment occurs.

If $R > \alpha$ then the *AIGRO* is negative. In this case the wealth effect is negative (i.e. it leads to a reduction of consumption and consequently to an increase in investment) while the substitution effect is positive. Therefore, if $\varepsilon < 1$ the wealth effect dominates the substitution effect leading to a reduction of consumption and an increase in investment. The opposite happens when $\varepsilon > 1$ and no change occurs if $\varepsilon = 1$.

When $R = \alpha$ the *AIGRO* does not change and therefore there will not be any effect on investment (for any value of ε).

Table 1 summarizes the sign between investment and uncertainty for different values of the parameters.

It is now possible to understand why in Nakamura (1999) the coefficient of relative risk aversion R is limited to assume values less than unity. In fact, as equation (36) shows and as Nakamura itself has pointed out (cfr. note 8 p.360) $\frac{dI}{d\sigma^2} > 0$ when $R > 1$, which is a very surprising result because it means that a very risk averse firm behaves like a risk-neutral firm. In our model this situation is equivalent to the case with $R > 1$ (which implies $\varepsilon < 1$ as $R = \frac{1}{\varepsilon}$

to have the traditional expected utility set-up): the separation between R and ε makes it easy to understand in this enviroment why $\frac{dI}{d\sigma^3} > 0$.

Nakamura also argues that when the coefficient of relative risk aversion is greater than one then the investment function has the unusual property that $\frac{dI}{d\delta} > 0$.[12] Using recursive preferences, this result can be easily explained too. Indeed from (31) we get

$$\frac{dI}{d\delta} = (1 - \varepsilon) K \tag{37}$$

which implies that $\frac{dI}{d\delta} \gtreqless 0$ as $\varepsilon \lesseqgtr 1$. The explanation is very similar to that given above concerning the relationship between uncertainty and investment. When the capital depreciation rate δ increases the rate of return on investment decreases. This has a negative wealth effect and a positive substitution effect.[13] If $\varepsilon < 1$ then the wealth effect prevails on the substitution effect and therefore consumption goes down and investment goes up. The opposit happens when $\varepsilon > 1$. This clarifies the Nakamura's result in two ways. First, the relationship between investment and capital depreciation rate is not influenced by the risk aversion but by the elasticity of intertemporal substitution. Second, we have a reasonable rationale of why $\frac{dI}{d\delta} > 0$ when $\varepsilon < 1$.[14]

Let's now analyse which of the above cases is the most likely given reasonable values of the preference parameters. The empirical results of Hall (1988) and of Campbell and Mankiw (1989) suggest a very small intertemporal substitution elasticity, perhaps on the order of 0.10. Therefore we can restrict our attention to the case with $\varepsilon < 1$.[15]

The likely level of relative risk aversion is probably more controversial but we can confidently assume R greater than α. Indeed, conventional estimates of R are in the range of 2 to 6. Kandel and Stambaugh (1991) argued that even values as high as 30 cannot be ruled out on the basis of available data. The Epstein and Zin (1991) estimate of R is around unity and is not grossly inconsistent with the hypothesis that R is as high as 2. In any case, if we consider $\alpha = 0.66$ it is quite reasonable to assume R greater than 0.66.

Under the above cited conditions ($R > \alpha$ and $\varepsilon < 1$) we know that an increase in uncertainty leads to an increase in investment level. This would suggest the existence of a positive relationship between investment and uncertainty which contradict the empirical evidence. However, the aim this paper is not to give a set-up where investment and uncertainty are negatively

[12] Again, see note 8 p.360.

[13] Clearly, the wealth and substitution effects are referred to consumption as before.

[14] Or better, we should say when $R > 1$ in the traditional set-np.

[15] All the others empirical studies about the elasticity of intertemporal substitution find values of ε significantly less than one.

related but to demonstrate that the intertemporal substitution elasticity is a key parameter in the explanation of the investment-uncertainty relationship. In particular, in this paper we want to show the exact role of risk-aversion and intertemporal substitution in the relationship between investment and uncertainty and emphasize that it is not true that an higher degree of risk-aversion does itself lead to a lower level of investment when uncertainty increases.

3 The investment with adjustment costs

We now assume that the production of the investment good requires adjustment costs $c(I)$ that we assume to be quadratic and equal to

$$c(I) = I + \frac{\lambda}{2} I^2 \tag{38}$$

All the others hypothesis still hold. The maximization problem of the present discounted value of the expected cash flow becomes:

$$V(\vartheta, K) = \max_{\{I\}} E_t \left\{ \int_t^{\infty} \exp\left[- \int_t^s \tilde{r}(u) \cdot du \right] \left[\vartheta^{\beta} K - c(I) \right] \cdot ds \right\} \tag{39}$$

The correspondent Bellman equation for this problem is

$$\tilde{r} V dt = \max_{\{I\}} \left\{ \left[\vartheta^{\beta} K - c(I) \right] \cdot dt + E_t(dV) \right\} \tag{40}$$

Using Ito's lemma to determine the expected capital gain $E_t(dV)$ and substituting it into equation (40) allows us to get

$$\tilde{r} V = \max_{\{I\}} \left\{ \vartheta^{\beta} K - c(I) + V_K I + \frac{1}{2} \sigma^2 \vartheta^2 V_{\vartheta\vartheta} \right\} \tag{41}$$

Maximizing (41) with respect to I leads to the first order condition $V_K = c'(I)$. Given the quadratic form of the adjustment costs, the latter becomes $q = 1 + \lambda I$, with $q \equiv V_K$. Now, by solving with respect to I we get the following investment function

$$I = \frac{1}{\lambda}(q - 1) \tag{42}$$

This investment function requires an explicit expression for q. If we substitute (42) into (41) we get

$$\tilde{r} V = \vartheta^{\beta} K - \frac{1}{2\lambda} q^2 + \frac{1}{2} \sigma^2 \vartheta^2 V_{\vartheta\vartheta} \tag{43}$$

In this case the firm is competitive, it has constant return to scale and adjustment costs do not depend on the capital stock. Under these conditions it is possible to show (cfr. Abel and Eberly (1994) and Abel and Eberly (1997)) that the fundamental value of the firm is a linear function of K and it has the following form:

$$V(\vartheta, K) = q(\vartheta) \cdot K + G(\vartheta) \tag{44}$$

where $G(\vartheta)$ equals the present value of the expected rents accruing to the adjustment technology represented by the convex cost of adjustment function.

Substituting (44) into (43) and solving for the fundamental value of the firm we obtain

$$V(\vartheta, K) = q(\vartheta) \cdot K + \frac{1}{2\lambda} \left[\frac{1}{\tilde{r}} - \frac{2\vartheta^{\beta}}{(\tilde{r} - m)^2} + \frac{\vartheta^{2\beta}}{(\tilde{r} - m)^2 (\tilde{r} - 2m - s^2)} \right] \tag{45}$$

where

$$q(\vartheta) = \frac{\vartheta^{\beta}}{\tilde{r} - m} = \frac{\vartheta^{\beta}}{\tilde{r} - \frac{\sigma^2}{2} \beta (\beta - 1)} \tag{46}$$

Equation (45) shows that the marginal value of the firm, $q(\vartheta)$, is equal to the discounted value of the expected marginal product of capital considering that the growth rate m of the value of the firm is substracted to the risky interest rate $\tilde{r}$.[16]

Therefore, the investment function (42) becomes

$$I = \frac{1}{\lambda} \left[\frac{\vartheta^{\beta}}{\tilde{r} - \frac{\sigma^2}{2} \beta (\beta - 1)} - 1 \right] \tag{47}$$

3.1 The effect of uncertainty on investment with adjustment costs

Let's now analyze the effect of an increase in uncertainty on investment:

$$\frac{dI}{d\sigma^2} = \frac{\partial I}{\partial \sigma^2} + \frac{\partial I}{\partial \tilde{r}} \frac{d\tilde{r}}{d\sigma^2} = \frac{\vartheta^{\beta}}{\lambda (\tilde{r} - m)^2} \left(\frac{dm}{d\sigma^2} - \frac{d\tilde{r}}{d\sigma^2} \right) \tag{48}$$

Equation (48) shows that the increase in uncertainty has two effects on investment. A positive *direct effect* $\left(\frac{\partial I}{\partial \sigma^2}\right)$ due to the fact that a greater uncertainty

[16] It is assumed that the value of the parameters in equation (46) are such that the denominator is always positive.

increases the instantaneous growth rate of output. An *indirect effect* $\left(\frac{\partial I}{\partial \check{r}}\frac{d\check{r}}{d\sigma^2}\right)$ originated by the variation of the risky rate of interest $\check{r}$ after the increase in uncertainty. The sign of the uncertainty-investment relationship depends on which of the two effects prevail. In particular, if the increase of the instantaneous growth rate of output $\left(\frac{dm}{d\sigma^2}\right)$ is greater than the increase of the risky rate of interest $\left(\frac{d\check{r}}{d\sigma^2}\right)$, then an increase in uncertainty leads to an higher level of investment and vice-versa. This consideration holds under the assumption that an increase in uncertainty leads to an higher risky rate of interest, i.e. $\frac{d\check{r}}{d\sigma^2} > 0$. However, although this result is quite reasonable, as we'll see, it cannot always be demonstrated. Needless to say that if the increase in uncertainty determines a lower risky rate of interest then the effect on investment is positive.

The rate of interest $\check{r}$ is determined by the following market equilibrium condition:[17]

$$\begin{aligned}\vartheta^\beta K &= C + I + \frac{\lambda}{2}I^2 = \\ &= \left\{\varepsilon\rho - (\varepsilon - 1)\left[r + \frac{(\check{r} - r)^2}{2Ru^2}\right]\right\}V + \\ &\quad \frac{1}{\lambda}\left(\frac{\vartheta^\beta}{\check{r} - m} - 1\right) + \frac{1}{2\lambda}\left(\frac{\vartheta^\beta}{\check{r} - m} - 1\right)^2 \end{aligned} \tag{49}$$

This condition can be reinterpreted in terms of equality between saving and adjustment costs which include the cost of purchase of capital goods. The introduction of the adjustment costs does not modify the portfolio allocation rule and the distribution of income between consumption and saving because the firm's rate of return is equal to $\check{r}$ as before. However, the presence of the adjustment costs implies that the consumer's wealth W is not equal to K anymore.

The effect of uncertainty on the risky interest rate can be determined from equation (49) using the implicit function theorem. Rewriting equation (49) as

$$IF \equiv \psi V + \frac{\vartheta^{2\beta}}{2\lambda(\check{r} - m)^2} - \frac{1}{2\lambda} - \vartheta^\beta K = 0 \tag{50}$$

[17] Notice that the stochastic component s^2 in the consumption function, which is identical to equation (27), is replaced by $u^2 = \beta^2\sigma^2\left[1 + \frac{G}{V} - \frac{\vartheta^\beta}{\lambda(\check{r}-m)^2 V}\right]^2$. Indeed, when the firm faces quadratic investment adjustment costs the consumer's maximization problem does not modify but the stochastic component of the budget coustraint. For further details see Appendix D.

and using the implicit function theorem we have

$$\frac{d\check{r}}{d\sigma^2} = -\frac{\frac{\partial IF}{\partial\sigma^2}}{\frac{\partial IF}{\partial\check{r}}} \tag{51}$$

where

$$\frac{\partial IF}{\partial\sigma^2} = (1-\varepsilon)\frac{d\check{r}}{d\sigma^2}V - (1-\varepsilon)\frac{1}{2}R\gamma V + \psi\frac{\partial V}{\partial\sigma^2} + \frac{\vartheta^{2\beta}}{\lambda(\check{r} - m)^3}\frac{1}{2}\beta(\beta - 1) \tag{52}$$

and

$$\frac{\partial IF}{\partial\check{r}} = (1-\varepsilon)V + \psi\frac{dV}{d\check{r}} - \frac{\vartheta^{2\beta}}{\lambda(\check{r} - m)^3} \tag{53}$$

From equation (45) we get

$$\begin{aligned}\frac{\partial V}{\partial\sigma^2} &= \frac{\vartheta^\beta K}{(\check{r} - m)^2}\frac{1}{2}\beta(\beta - 1) + \\ &\quad \frac{\vartheta^{2\beta}}{\lambda(\check{r} - m)^3(\check{r} - 2m - s^2)}\frac{1}{2}\beta(\beta - 1) + \\ &\quad + \frac{\vartheta^{2\beta}}{2\lambda(\check{r} - m)^2(\check{r} - 2m - s^2)^2}\beta(2\beta - 1) - \frac{2\vartheta^\beta}{\lambda(\check{r} - m)^3}\frac{1}{2}\beta(\beta - 1)\end{aligned} \tag{54}$$

and

$$\begin{aligned}\frac{\partial V}{\partial\check{r}} &= -\frac{\vartheta^\beta K}{(\check{r} - m)^2} - \frac{\vartheta^{2\beta}}{\lambda(\check{r} - m)^3(\check{r} - 2m - s^2)} - \\ &\quad - \frac{\vartheta^{2\beta}}{2\lambda(\check{r} - m)^2(\check{r} - 2m - s^2)^2} + \\ &\quad - \frac{1}{2\lambda\check{r}^2} + \frac{2\vartheta^\beta}{\lambda(\check{r} - m)^3}\end{aligned} \tag{55}$$

Substituting equations (52), (53), (54) and (55) into (51) and simplifying we obtain

$$\frac{d\check{r}}{d\sigma^2} = \frac{(\varepsilon - 1)\frac{1}{2}R\gamma V + \psi\frac{\partial V}{\partial\sigma^2} + \frac{\vartheta^{2\beta}}{\lambda(\check{r}-m)^3}\frac{1}{2}\beta(\beta - 1)}{2(1-\varepsilon)V + \psi\frac{dV}{d\check{r}} - \frac{\vartheta^{2\beta}}{\lambda(\check{r}-m)^3}} \tag{56}$$

Plugging (56) into (48) leads to

$$\begin{aligned}\frac{dI}{d\sigma^2} &= \frac{\vartheta^\beta}{\lambda(\check{r} - m)^2} - \frac{(\varepsilon - 1)V\left[\beta(\beta - 1) - \frac{1}{2}R\gamma\right]}{2(\varepsilon - 1)V - \psi\frac{dV}{d\check{r}} + \frac{\vartheta^{2\beta}}{\lambda(\check{r}-m)^3}} \\ &\quad \frac{\psi\left[\frac{\vartheta^{2\beta}}{2\lambda(\check{r}-m)^2(\check{r}-2m-s^2)^2}\frac{1}{2}\beta(3\beta - 1) - \frac{1}{4\lambda\check{r}^2}\beta(\beta - 1)\right]}{2(\varepsilon - 1)V - \psi\frac{dV}{d\check{r}} + \frac{\vartheta^{2\beta}}{\lambda(\check{r}-m)^3}}\end{aligned} \tag{57}$$

The presence of adjustment costs makes the investment-uncertainty relationship more complicated because the consumer's decisions depend also from their wealth W which is not exogeneous and equal to the capital stock K anymore. In particular, the consumer's wealth is equal to the value of the firm and when uncertainty increases, V changes both because of the change of σ^2 and the risky rate of interest $\check{r}$. Equation (57) shows that the sign of the relationship between investment and uncertainty is always ambiguous.

4 Irreversibile investment

To see if it is possible to have more information about the elements that determine the sign of the investment-uncertainty relationship when there are adjustment costs, we consider the case where the price of capital is zero and the firm has to pay quadratic adjustment costs to invest (as in Abel (1983)), which are given by

$$c(I) = \frac{\lambda}{2} I^2 \tag{58}$$

This implies the existence of irreversibility because it is never convenient to disinvest. Repeating the same procedure of the previous paragraph we obtain that the fundamental value of the firm is

$$V(\vartheta, K) = \frac{\vartheta^\beta K}{\check{r} - m} + \frac{\vartheta^{2\beta}}{2\lambda (\check{r} - m)^2 (\check{r} - 2m - s^2)} \tag{59}$$

and the investment function

$$I = \frac{\vartheta^\beta}{\lambda (\check{r} - m)} \tag{60}$$

The maximization problem of the consumer does not change and the consumption function is still given by equation (27)

$$C = \psi V$$

where ψ is the same of equation (23).[18]

As before, the effect of uncertainty on investment is given by

$$\frac{dI}{d\sigma^2} = \frac{\partial I}{\partial \sigma^2} + \frac{\partial I}{\partial \check{r}} \frac{d\check{r}}{d\sigma^2} = \frac{\vartheta^\beta}{\lambda(\check{r} - m)^2} \left(\frac{dm}{d\sigma^2} - \frac{d\check{r}}{d\sigma^2} \right) \tag{61}$$

which is exactly equal to equation (48).

[18]However, the stochastic component in this context is equal to $u^2 = \beta^2 \sigma^2 \left(1 + \frac{G}{V}\right)^2$.

The market equilibrium condition that determines $\check{r}$ is

$$\vartheta^\beta K = C + \frac{\lambda}{2} I^2 \tag{62}$$

and this implies that equation (50) becomes

$$IF \equiv \psi V + \frac{\vartheta^{2\beta}}{2\lambda (\check{r} - m)^2} - \vartheta^\beta K = 0 \tag{63}$$

As in the previous problem

$$\frac{d\check{r}}{d\sigma^2} = - \frac{\frac{\partial IF}{\partial \sigma^2}}{\frac{\partial IF}{\partial \check{r}}} \tag{64}$$

From equation (63) it follows that

$$\frac{\partial IF}{\partial \sigma^2} = (1 - \varepsilon) \frac{d\check{r}}{d\sigma^2} V - (1 - \varepsilon) \frac{1}{2} R\gamma V + \psi \frac{\partial V}{\partial \sigma^2} + \frac{\vartheta^{2\beta}}{\lambda (\check{r} - m)^3} \frac{1}{2} \beta (\beta - 1) \tag{65}$$

and

$$\frac{\partial IF}{\partial \check{r}} = (1 - \varepsilon) V + \psi \frac{dV}{d\check{r}} - \frac{\vartheta^{2\beta}}{\lambda (\check{r} - m)^3} \tag{66}$$

Equation (59) implies that

$$\begin{aligned} \frac{\partial V}{\partial \sigma^2} &= \frac{\vartheta^\beta K}{(\check{r} - m)^2} \frac{1}{2} \beta (\beta - 1) + \frac{\vartheta^{2\beta}}{\lambda (\check{r} - m)^3 (\check{r} - 2m - s^2)} \frac{1}{2} \beta (\beta - 1) + \\ &+ \frac{\vartheta^{2\beta}}{2\lambda (\check{r} - m)^2 (\check{r} - 2m - s^2)^2} \beta (2\beta - 1) \end{aligned} \tag{67}$$

and

$$\begin{aligned} \frac{\partial V}{\partial \check{r}} &= - \frac{\vartheta^\beta K}{(\check{r} - m)^2} - \frac{\vartheta^{2\beta}}{\lambda (\check{r} - m)^3 (\check{r} - 2m - s^2)} + \\ &\quad \frac{\vartheta^{2\beta}}{2\lambda (\check{r} - m)^2 (\check{r} - 2m - s^2)^2} \end{aligned} \tag{68}$$

The substitution of equations (65), (66), (67) and (68) into (64) and some simplifications gives

$$\frac{d\check{r}}{d\sigma^2} = - \frac{(\varepsilon - 1) \frac{1}{2} R\gamma V + \psi \frac{\partial V}{\partial \sigma^2} + \frac{\vartheta^{2\beta}}{\lambda (\check{r} - m)^3} \frac{1}{2} \beta (\beta - 1)}{2 (1 - \varepsilon) V + \psi \frac{dV}{d\check{r}} - \frac{\vartheta^{2\beta}}{\lambda (\check{r} - m)^3}} \tag{69}$$

$\frac{dI}{d\sigma^2}$	$R > 2\alpha$	$R = 2\alpha$	$R < 2\alpha$
$\varepsilon < 1$	$\gtrless 0$	$\gtrless 0$	$\gtrless 0$
$\varepsilon = 1$	< 0	< 0	< 0
$\varepsilon > 1$	< 0	$\gtrless 0$	$\gtrless 0$

Table 2:

Plugging (69) into (61) leads to

$$\frac{dI}{d\sigma^2} = \frac{\vartheta^\beta}{\lambda(\check{r}-m)^2}\left\{\frac{(\varepsilon-1)\,V\left[\beta\,(\beta-1)-\frac{1}{2}R\gamma\right]}{2\,(\varepsilon-1)\,V-\psi\frac{dV}{d\check{r}}+\frac{\vartheta^{2\beta}}{\lambda(\check{r}-m)^3}} \right. \\ \left. \frac{-\psi\frac{\vartheta^{2\beta}}{2\lambda(\check{r}-m)^2(\check{r}-2m-s^2)^2}\frac{1}{2}\beta\,(3\beta-1)}{2\,(\varepsilon-1)\,V-\psi\frac{dV}{d\check{r}}+\frac{\vartheta^{2\beta}}{\lambda(\check{r}-m)^3}}\right\} \tag{70}$$

Equation (70) shows that the intertemporal elasticity substitution is still a key parameter in the determination of the sign in the uncertainty-investment relationship. Let's first consider the case with $\varepsilon > 1$: the denominator of equation (70) has always a positive sign.[19] If $R \geq 2\alpha$ then $\left[\beta\,(\beta-1)-\frac{1}{2}R\gamma\right] \leq 0$ and the numerator is negative.[20] Therefore, it follows that $\frac{dI}{d\sigma^2} < 0$ as in the reversible investment case. When $R < 2\alpha$ then $\left[\beta\,(\beta-1)-\frac{1}{2}R\gamma\right] > 0$ and the sign of the investment-uncertainty relationship becomes ambiguous.

Consider now the case with $\varepsilon < 1$: the relationship between investment and uncertainty has not a definite sign. Indeed, when $\varepsilon < 1$ the sign of both the denominator and the numerator is ambiguous.

When $\varepsilon = 1$ then the denominator is positive, the numerator is always negative and it follows that $\frac{dI}{d\sigma^2} < 0$. This is the difference with the reversible investment set-up where $\frac{dI}{d\sigma^2}$ is always zero when $\varepsilon = 1$: the presence of adjustment costs and irreversibility implies a negative relationship between investment and uncertainty.

Table 2 summarizes the sign between investment and uncertainty for different valnes of the parameters.

Comparing the results obtained in this framework with those obtained without adjustment costs we can make the following two considerations. First, when there is irreversibility and adjustment costs the uncertainty-investment relationship has a not in general a definite sign. Second, the only case where we have a definte sign is when the intertemporal substitution elasticity is one: the presence of irreversibility (and adjustment costs) push towards a negative relationship between investment and uncertainty.

[19]Notice that equation (68) implies that $\frac{\partial V}{\partial \check{r}}$ is always negative.

[20]This condition is obtained assuming $\gamma \cong \beta^2$ which is a good approximation when we are close to the steady-state.

5 Conclusion

In this paper we have analyzed the relationship between uncertainty and investment when firms are risk averse and have a constant returns to scale technology. We have done it using a general equilibrium model where firms decide the optimal investment and consumers choose their consumption an portfolio allocation. Using recursive preferences that allow us to separate the relative risk aversion from the intertemporal substitution elasticity, we demonstrate that the role of the risk aversion in the investment-uncertainty relationship is not necessarily negative, as argued by the traditional literature, but it depends from the degree of intertemporal substitutability. In particular, when the risk aversion is high and the intertemporal substitution is low (which is the case that we have when we assign reasonable values to the parameters) then an increase in uncertainty increases the level of investment.

In the second part of the paper we have introduced the adjustment costs . In this framework, the uncertainty-investment relationship has a not a definite sign. However, if the intertemporal substitution elasticity is equal to one and there is irreversibility (in the sense of Abel (1983)), then the relation analyzed becomes negative. Notice that without adjustment costs and with a unitary intertemporal substitution elasticity, the uncertainty has no effect on investment. Therefore, we can say that adjustment costs and irreversibility have a negative effect on investment when uncertainty increases.

Appendix A

In this appendix we show that the only economically relevant solution for the value of the firm is $V = \frac{\vartheta^\beta K}{\check{r}-m}$. Equation (8) is an ordinary second order differential equation in ϑ with variable coefficients

$$\check{r}V = \vartheta^\beta K + \frac{1}{2}\sigma^2\vartheta^2 V_{\vartheta\vartheta}$$

We know that the general solution of this equation is given by the sum of any particular solution of the complete equation and the general solution of the reduced equation. Using the variation of parameters method we obtain the general solution

$$V = A_1\vartheta^{n_1} + A_2\vartheta^{n_2} + \frac{\vartheta^\beta K}{\tilde{r} - m} \tag{A.1}$$

where n_1 and n_2 are the solutions of the following characteristic equation of (8)

$$\zeta(n) = \frac{1}{2}\sigma^2 n^2 - n\frac{\sigma^2}{2} - \tilde{r} \tag{A.2}$$

In particular,

$$n_1, n_2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2\tilde{r}}{\sigma^2}} \tag{A.3}$$

with $n_1 > 0$ and $n_2 < 0$.

The term $\frac{\vartheta^\beta K}{\tilde{r}-m}$ of the equation (A.1) is the expected discounted value of the flows of output deriving from the use of capital. The term $\frac{1}{\tilde{r}-m}$ is a discounting factor: $\tilde{r}$ is the rate of interest required by the market and m is the expected growth rate of output. The particular solution is the fundamental of the value of the firm. The other two general solutions of the reduced equation in (A.1) can be eliminated because of the following considerations.

High values of ϑ should increase the value of the firm because a positive technological shock increases the output and the operating profit $\vartheta^\beta K$. The solution $A_2\vartheta^{n_2}$ implies that the value of the firm decreases when ϑ increases because $n_2 < 0$. Therefore, it can be assumed that $A_2 = 0$.

The solution $A_1\vartheta^{n_1}$ can be considered a speculative component. If the value of the firm would be determined from $A_1\vartheta^{n_1}$, then the expected instantaneous capital gains when no investment are done are given by

$$\begin{aligned} \frac{1}{dt}\frac{E_t(dV)}{V} &= \frac{1}{dt}\frac{1}{V}E_t\left[A_1 n_1\vartheta^{n_1-1}d\vartheta + \frac{1}{2}A_1 n_1(n_1-1)\vartheta^{n_1-2}(d\vartheta)^2\right] \\ &= \frac{\sigma^2}{2}n_1(n_1-1) = \tilde{r} \end{aligned} \tag{A.4}$$

where the last equality comes from equation (A.2). If the value of the firm is given by (A.4) then the capital gain would itself be sufficient to guarantee an instantaneous rate of return equal to $\tilde{r}$. However, this implies the violation of the no arbitrage condition. Indeed, an individual could buy the firm paying a rate of interest $\tilde{r}$, get dividends $\vartheta^\beta K$ and sell the firm obtaining an expected capital gain equal to $\tilde{r}$. This strategy allows to get dividends at zero cost. Therefore the no arbitrage condition implies $A_1 = 0$.

Appendix B

In this appendix we show how to obtain the budget constraint (15). We start from

$$dV = V_K \cdot dK + V_\vartheta \cdot d\vartheta + \frac{1}{2}V_{\vartheta\vartheta}\cdot(d\vartheta)^2 \tag{B.1}$$

From equation (9) we can determine

$$V_\vartheta = \beta\frac{\vartheta^{\beta-1}K}{\tilde{r}-m} \tag{B.2}$$

$$V_{\vartheta\vartheta} = \beta(\beta-1)\frac{\vartheta^{\beta-2}K}{\tilde{r}-m} \tag{B.3}$$

Substituting equation (B.2), (B.3), the first order condition (7) and the capital accumulation constraint $dK = Idt$ into (B.1) we obtain

$$dV = Idt + \beta\frac{\vartheta^{\beta-1}K}{\tilde{r}-m}d\vartheta + \frac{1}{2}\beta(\beta-1)\frac{\vartheta^{\beta-2}K}{\tilde{r}-m}(d\vartheta)^2 \tag{B.4}$$

Considering the stochastic process (1) of ϑ, equation (B.1) can be rewritten as

$$dV = Idt + \beta\sigma\frac{\vartheta^\beta K}{\tilde{r}-m}dz + \frac{1}{2}\beta(\beta-1)\sigma^2\frac{\vartheta^\beta K}{\tilde{r}-m}(dz)^2 \tag{B.5}$$

From (B.5) we have that the expected capital gain is

$$\frac{1}{dt}\frac{E_t(dV)}{V} = \frac{I}{V} + m \tag{B.6}$$

and the instantaneous variance

$$\frac{1}{dt}\frac{var(dV)}{V} = \beta^2\sigma^2 \equiv s^2 \tag{B.7}$$

Therefore the stochastic process of the value of the firm becomes

$$\frac{dV}{V} = \left(\frac{I}{V} + m\right)\cdot dt + s\cdot dz \tag{B.8}$$

The consumers get the capital gain (dV/V) and the instantaneous flow of dividend $(\vartheta^\beta K - I)$. It follows that the budget constraint

$$dW = W\left[a\left(\frac{dV}{V} + \frac{\vartheta^\beta K - I}{V}dt\right) + (1-a)\,rdt\right] - Cdt \tag{B.9}$$

can be written as

$$dW = [a\check{r}W + (1-a)\,rW - C]\cdot dt + aWu\cdot dz$$

which is exactly equation (15).

Appendix C

To see why ν is a risk-adjusted discount rate, let's consider the problem defined by the following Bellman equation:

$$\begin{aligned}\nu\cdot f\left[(1-R)\,J\,(W)\right] &= \max_{a,C}\left\{\frac{1-R}{1-\frac{1}{\varepsilon}}C^{1-\frac{1}{\varepsilon}} + (1-R)\cdot f'\left[(1-R)\,J\,(W)\right]\right. \qquad (C.1)\\ &\cdot\left.\left\{J'\,(W)\left[a\check{r}W + (1-a)\,rW - C\right] + \frac{1}{2}J''\,(W)\,a^2s^2W^2\right\}\right\}\end{aligned}$$

Differently from the problem examined in the main text, in the place of ρ we have ν as a discount rate. However, the two problems have the same solution.

To verify this, derive the first order conditions:

$$J'\,(W)\,(\check{r}-r) + J''\,(W)\,as^2W = 0$$

$$C^{-\frac{1}{\varepsilon}} - f'\left[(1-R)\,J\,(W)\right]J'\,(W) = 0$$

which are exactly equation (17) and (18) derived before in the main problem. Then make the same guess for the value function J, i.e. equation (19), and substitute $J\,(W)$, $J'\,(W)$ and $J''\,(W)$ in the first order conditions. This leads to:

$$a = \frac{\check{r}-r}{Ru^2}$$

and

$$C = b^{1-\varepsilon}W$$

Next, consider the undetermined constant $\psi \equiv b^{1-\varepsilon}$. Substituting $J\,(W)$, $J'\,(W)$ and $J''\,(W)$ and the above equations in the Bellman equation (C.1), we get

$$\psi = \varepsilon\rho - (\varepsilon - 1)\left[r + \frac{(\check{r}-r)^2}{2Rs^2}\right]$$

which again is the same solution derived in the main text (see equation (23)).

Appendix D

In this Appendix we show that the stochastic component in the consumer's problem is

$$u = \beta\sigma\left[1 + \frac{G}{V} - \frac{\vartheta^\beta}{\lambda(\check{r}-m)^2V}\right]$$

We start from equation (B.1):

$$dV = V_K\cdot dK + V_\vartheta\cdot d\vartheta + \frac{1}{2}V_{\vartheta\vartheta}\cdot(d\vartheta)^2$$

where

$$\begin{aligned}V\,(\vartheta, K) &= q\,(\vartheta)\cdot K + G\,(\vartheta)\\ &= \frac{\vartheta^\beta}{\check{r}-m}K + \frac{1}{2\lambda}\left[\frac{1}{\check{r}} - \frac{2\vartheta^\beta}{(\check{r}-m)^2} + \frac{\vartheta^{2\beta}}{(\check{r}-m)^2\,(\check{r}-2m-s^2)}\right]\end{aligned}$$

Therefore

$$V_\vartheta = \beta\frac{\vartheta^{\beta-1}K}{\check{r}-m} + 2\beta\frac{\vartheta^{2\beta-1}}{2\lambda\,(\check{r}-m)^2\,(\check{r}-2m-s^2)} - \beta\frac{\vartheta^{\beta-1}}{\lambda\,(\check{r}-m)^2}$$

$$\begin{aligned}V_{\vartheta\vartheta} &= \beta\,(\beta-1)\,\frac{\vartheta^{\beta-2}K}{\check{r}-m} + \\ &\quad +2\beta\,(2\beta-1)\,\frac{\vartheta^{2\beta-2}}{2\lambda\,(\check{r}-m)^2\,(\check{r}-2m-s^2)} - \beta\,(\beta-1)\,\frac{\vartheta^{\beta-2}}{\lambda\,(\check{r}-m)^2}\end{aligned}$$

Following the same procedure of Appendix B we have that the stochastic component is

$$\begin{aligned}sc &= \left[\beta\frac{\vartheta^{\beta-1}K}{\check{r}-m} + 2\beta\frac{\vartheta^{2\beta-1}}{2\lambda\,(\check{r}-m)^2\,(\check{r}-2m-s^2)} - \beta\frac{\vartheta^{\beta-1}}{\lambda\,(\check{r}-m)^2}\right]\sigma\vartheta dz\\ &= \beta\sigma\left[V + G - \frac{\vartheta^\beta}{\lambda(\check{r}-m)^2}\right]dz\end{aligned}$$

Imposing

$$\beta\sigma\left[V + G - \frac{\vartheta^\beta}{\lambda(\check{r}-m)^2}\right] = uV$$

we obtain

$$u = \beta\sigma\left[1 + \frac{G}{V} - \frac{\vartheta^\beta}{\lambda(\check{r}-m)^2V}\right]$$

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